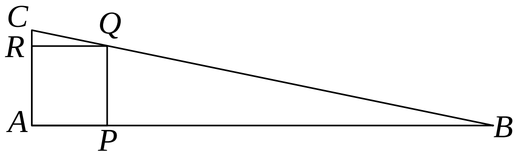
2010 Vector Article A:

Problem 1: In the picture below, ΔABC is right-angled at A, P lies on AB, Q lies on BC, R lies on CA, and $APQR$ is a square. The length of AB is 24 and the length of AC is 5. What is the length of AP?



Solution: In almost any problem involving a triangle where a line is drawn parallel to one of its sides, the fact that some triangles are similar is useful. Let $x = AP = PQ$. Notice that ΔPBQ and ΔABC are similar (angles match, the first triangle is a scaled version of the second triangle). Thus

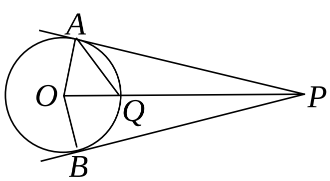
$$\frac{AC}{PQ} = \frac{AB}{PB}$$

But, $AC = 5$, $AB = 24$, $PQ = x$, and $PB = 24 - x$. Thus,

$$\frac{5}{x} = \frac{24}{24 - x}\ .$$

Multiply by $x(24 - x)$ to get $120 - 5x = 24x$, and simplified to $120 = 29x$. Solve: $x = \dfrac{120}{29}$.

Problem 2: The diagram shows a circle, and two tangent lines PA and PB. The points A, B, and Q are on the circle, and Q is on the line segment that joins the centre O of the circle to P. Suppose that the measure of $\angle APB$ is $34°$. What is the degree measure of $\angle AQB$? (In the original problem, line segments OA and OB were not drawn.)

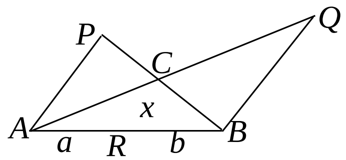


Solution: In almost any problem involving circles, it is useful to put in the centres of the circles. Join this centre O to the two points of tangency. OP bisects $\angle APB$. Thus, $\angle OPA$ is $17°$, and $\angle OAP$ is $90°$. It follows that $\angle AOP$ is $73°$. That leaves $107°$ to be distributed between $\angle OAQ$ and $\angle OQA$. Since OA and OQ are radii, $\angle OQA = \angle OAQ = 107/2$. By symmetry of triangles, $\angle OQA = \angle OQB$. Therefore, $\angle AQB = \angle OQA + \angle OQB = 107°$.

Comment: In order to make the problem easier, the point Q was put exactly midway between A and B. But move it a little, to a point Q'. Let $x = \angle OQ'A$ and let $y = \angle OQ'B$. A calculation much like the one above shows that $\angle OAQ' = (90 + x)/2$ and $\angle OBQ' = (90 + y)/2$. Thus $\angle AQ'B = 90 + (x + y)/2$. Since $x + y = 34$, we conclude that $\angle AQ'B = 90 + 34/2 = 107°$.

Problem 3: Consider all ordered pairs (A, B), where A and B are distinct vertices of a regular hexagon whose edges have length 5. What is the average value of the distance between A and B?
Solution: In a regular polygon the sides are the same length and the angles all have the same measure. Thus, by symmetry, the average is the same as the average if A is any specified vertex. So, now A is fixed, and B can take on 5 values. Going clockwise from A, the first distance and the fifth distance are each equal to 5. The second and the fourth distances are a bit harder to compute. But, using the fact that in a $30-60-90$ triangle the hypotenuse is twice the length of the smallest side, we find that these are each $2\times 5\sqrt{3}/2$, or $5\sqrt{3}$. The third distance is easily determined as 10. Just think of the circle that circumscribes the hexagon: A and the third vertex from A form a diameter of the circle whose radius is 5. Add up the 5 distances 5, 5, $5\sqrt{3}$, $5\sqrt{3}$, and 10. The average is the sum divided by 5. Thus, we get
$(5+5+5\sqrt{3}+5\sqrt{3}+10)/5=(20+10\sqrt{3})/5=4+2\sqrt{3}$.

Problem 4: In the picture, ABC is a triangle, and R is a point on the line segment AB. The point P is on BC extended, with AP parallel to RC. Similarly, Q lies on AC extended, with BQ parallel to RC. Given that $AP=5$ and $BQ=6$, what is the length of the line segment RC? (In the original problem, labels x, a, and b were not given.)



Solution: Again, we are going to use the fact that some triangles are similar. Let $x=RC$, let $a=AR$, and let $b=BR$. Because ΔPAB and ΔCRB are similar,
$\frac{x}{5}=\frac{b}{a+b}$.
Also, ΔQAB and ΔCAR are similar, so
$\frac{x}{6}=\frac{a}{a+b}$.
Add the above. Thus, $\frac{x}{5}+\frac{x}{6}=\frac{b}{a+b}+\frac{a}{a+b}=\frac{a+b}{a+b}=1$.
Multiply the left side and the right side by 30: $6x+5x=30$. Solve: $x=\frac{30}{11}$.

2011 Vector Magazine Article A:

Problem 1:

You play the "Lucky 4" game as follows. You roll a fair standard die and receive in pennies the number you rolled. You keep doing this until either you have accumulated exactly 4 pennies, in which case you win, and the game is over, or your last roll pushes you over 4 pennies, in which case you lose. What is the probability that you win? Express the answer as a common fraction.

Solution.

Maybe you are *very* lucky, and immediately toss a 4 (probability: 1/6). Game over, you have won. You also win if you toss a 3, then a 1 (probability: $1/6 \times 1/6 = (1/6)^2$); or a 1, then a 3 (probability: $(1/6)^2$); or a 2, then a 2 (probability: $(1/6)^2$).

You can also win with 2, then 1, then 1; or with 1, 2, 1;
or with 1, 1, 2 (each has probability $(1/6)^3$).

Finally, you win the least likely way if you toss 1, 1, 1, 1 (probability $(1/6)^4$).

Add up, we get $1/6 + 3(1/6)^2 + 3(1/6)^3 + (1/6)^4$, which "simplifies" to $\frac{343}{1296}$.

Another way.

The final answer turns out to be $7^3/6^4$. Such a nice-looking answer makes one suspect there may be a more structured approach.

What is the probability of winning in *exactly* k tosses? It is not hard to see that this is $(1/6)^k$ times the number of ways that we can get a sum of 4 in exactly k tosses.

Let $C(n,r)$ be the number of ways of choosing r objects from n distinct objects. Now, imagine putting 4 identical doughnuts in a row. Then there are 3 "inter doughnut" gaps. The number of ways to get a total of 4 in exactly k tosses is the same as the number of ways of putting $k-1$ "dividers" in the 3 gaps (each of the 3 gaps is allowed to have at most one divider). The number of doughnuts up to the first divider represents the result of the first toss, the number of doughnuts between the first two dividers represents the result of the second toss, and so on. The same idea shows that the number of ways of distributing n identical doughnuts between k people so that everyone gets at least one doughnut is $C(n-1, k-1)$.

Thus the number of ways of winning in one toss is $C(3,0)$, the number of ways of winning in two toss is $C(3,1)$, and so on. Thus the probability of winning in Lucky 4 is
$C(3,0)\times(1/6) + C(3,1)\times(1/6)^2 + C(3,2)\times(1/6)^3 + C(3,3)\times(1/6)^4$.

Take out the common factor $1/6$. We are left with the binomial expansion of $(1+\frac{1}{6})^3$, so the required probability is $(1/6)(7/6)^3$.

Essentially the same argument shows that the probability of winning in the "Lucky 5" game is $(1/6)(7/6)^4$, and that the probability of winning "Lucky 6" is $(1/6)(7/6)^5$. These would be much lengthier to deal with using our first approach. A small modification works for "Lucky 7". Beyond 7, things get more complicated.

Problem 2:
What is the smallest positive integer n such that the leftmost digit in the decimal representation of 2^n is equal to 7? Hint: $2^{10} = 1024$.
Solution.
The first ten powers of 2, starting with the 0-th power, are 1, 2, 4, 8, 16, 32, 64, 128, 256, and 512. The next ten powers of 2 are 2^{10}, 2^{11}, and so on up to 2^{19}. These are 1×1024, 2×1024, 4×1024, and so on up to 512×1024.
And the next ten powers of 2 are 2^{20} to 2^{29}. These are $1 \times (1024)^2$, $2 \times (1024)^2$, and so on up to $512 \times (1024)^2$.
Continue: for each successive collection of ten powers of 2, we multiply the numbers 1, 2, 4, 8, $\cdots$, 512 by a suitable power of 1024. These powers of 1024, for a while, have a decimal expansion that has shape $(1.\ldots) \times (10)^j$. So if n is small, multiplying K by $(1024)^n$ does nothing dramatic to the leftmost digit of the decimal expansion of K.
We want to "push" one of the numbers 1 to 512 into having a decimal expansion that begins with 7, by multiplying by a suitable power of 1024. The number 64 requires the smallest push so that the leftmost digit is 7. More precisely, note that $70/64 = 1.09375$, so the needed push should be just under 1.1.
Experiment: 1024 is definitely not enough of a push; neither is $(1024)^2$. It is a bit less obvious that $(1024)^3$ is not enough. We need to check that $64 \times (1024)^3$ still starts with the digit 6. This is not too hard. No calculators were allowed, so the work can be delegated to a team member: $(1.024)^2$ is approximately 1.05, and (1.05)(1.024) is still well under 70/64. But $(1.024)^4$ is about 1.1, and we need to increase 6.4 by only around 9.4% to get to 7. So the first power of 2 that has the leftmost digit 7 is $64 \times (1024)^4$, that is, $2^6 \times 1024^4 = 2^{46}$, and $n = 46$. (It takes even longer to get a power of 2 that starts with 9. We can in fact find a power of 2 whose decimal expansion starts with any specified list of d digits.)

Problem 3:
There is a group of 7 women and m men arranged around a circular table so that the number of people whose right-hand neighbour is of the same sex is the same as the number of people whose right-hand neighbour is of the opposite sex. What is the largest possible value of m?
Solution.
It helps to draw a little sketch while following the argument. First of all, imagine that women *A* and *B* are next to each other (i.e. no men sit between them), and that the number of people whose right-hand neighbour is of the same sex is the same as the number of people whose right-hand neighbour is of the opposite sex. Now, put four new men between *A* and *B*. It is easy to see that the number of people whose right-hand neighbour is of the same sex has increased by 2, and the number of people whose right-hand neighbour is of the opposite sex has also increased by 2. Thus, if we have a seating configuration that satisfies the condition in which there are no men between *A* and *B*, then adding 4 more men between *A* and *B* is also a valid configuration in which the number of men is increased by 4. Since we want to maximize the number of men, we can assume from now on that no two women are next to each other.
If no two women are next to each other, the number of people whose right-hand neighbour is of the opposite sex is 14 (all the women, and all their male left-hand neighbours). Let $m_1, m_2, \ldots, m_7$ be

the number of men in the 7 gaps between women. Since no two women are next to each other, the number of people whose right-hand neighbour is of the same sex is

$(m_1 - 1) + (m_2 - 1) + \cdots + (m_7 - 1)$.

Set this equal to 14. We find that $m_1 + m_2 + \cdots + m_7 = 21$, so the maximum number of men is 21. (Exactly the same argument shows that if there are w women, the maximum possible number of men is $3w$.)

Problem 4:

Let $A(n) = \frac{n(3n-1)}{2}$. What is the smallest integer $n > 1$ such that $A(n)$ is a perfect square?

Solution.

It turns out that $n = 81$, so if a team's search is suitably organized, there is a fair chance of finding the answer in time. But the search territory can be cut down a lot. Let $A(n)$ be the perfect square x^2. Then $n(3n-1) = 2x^2$. Since any prime factor of n is also a factor of $3n$ then it is clear that n and $3n-1$ can have no prime factor in common. So, either (i) n is a perfect square and $3n-1$ is twice a perfect square or (ii) n is twice a perfect square and $3n-1$ is a perfect square.

In case (i), $n = k^2$, and we want $3k^2 - 1$ to be twice a perfect square. This forces k to be odd. So try $k = 3$, $k = 5$, and so on. Quickly, we find that $k = 9$ works, since then $3k^2 - 1 = 242 = 2(11)^2$.

In the meantime, perhaps, another team member can be dealing with case (ii). There n is twice a perfect square, say $n = 2k^2$, so we want $3n - 1 = 6k^2 - 1$ to be a perfect square. One can search; but the search will be futile, for $3n-1$ can never be a perfect square. The reason is that $3n-1$ leaves a remainder of 2 on division by 3. But a perfect square always leaves a remainder of 0 or 1 on division by 3. (Any integer a is of the form $3t$, $3t+1$, or $3t+2$, where t is an integer. Clearly, $(3t)^2$ is divisible by 3. By expanding $(3t+1)^2$ and $(3t+2)^2$ we can see that each is 1 more than a multiple of 3.)

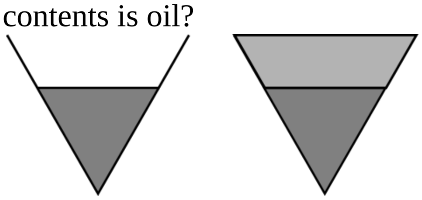
2012 Vector Magazine Article A:
Problem 1:
A line passes through the points $(-1,10)$, $(10,-1)$, and $(x,-10)$. What is the value of x?
Solution:
Draw a picture on a grid. As we travel along the line from $(-1,10)$, to $(10,-1)$, the x-coordinate increases by 11, and the y-coordinate decreases by 11. As we travel further to $(x,-10)$, the y-coordinate decreases by a further 9, so the x-coordinate increases by 9, to 19.

Equivalently, the slope (m) of the line is $\frac{10-(-1)}{-1-(10)}$, that is, $m=-1$. Since the points lie on a line, it follows that $\frac{10-(-10)}{-1-(x)}=-1$. A little manipulation gives $x=19$.

Or else note that because of the symmetry, the line has equation of the shape $x+y=k$. But $(-1,10)$ is on the line, so $k=9$. Thus when the y-coordinate is -10, $x+(-10)=9$, so the x-coordinate must be 19.

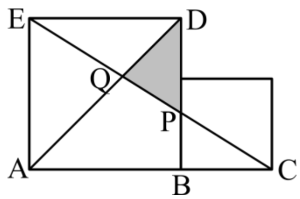
Or else recall that the line has an equation of the shape $y=mx+b$. Thus $10=-m+b$ and $-1=10m+b$. We can solve for m and b. Once we know the equation of the line, the problem is essentially solved.

Problem 2:
Vinegar is poured into a conical cup of height 3 inches until the vinegar is 2 inches deep at its deepest point (please see the left-hand diagram). Then olive oil is poured into the cup until the cup is full (right-hand diagram). After the oil has been poured in, what common fraction of the cup's contents is oil?



Solution:
We do not need to recall the formula for the volume of a cone: everything can be done by using the powerful idea of scaling. The vinegar cone is a scaled down version of the full cone, with all full cone dimensions scaled down by the scaling factor $\frac{2}{3}$ (the ratio of the heights of the liquid in the two cones). Thus volume is scaled down by the scaling factor $\left(\frac{2}{3}\right)^3$, and the volume of the vinegar is $\frac{8}{27}$ times the volume of the full cone. It follows that the volume of the oil is $1-\frac{8}{27}=\frac{19}{27}$ times the volume of the full cone.

Problem 3:
The larger square in the diagram has side 5 and the smaller square has side 3. What is the area of the shaded triangle? Express the answer as a common fraction.



Solution:

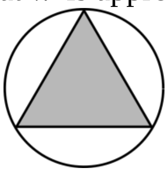
The solution makes heavy use of similarity. Triangles PBC and EAC are similar. Thus $\frac{PB}{3}=\frac{5}{8}$, so $PB=\frac{15}{8}$. It follows that $PD=5-\frac{15}{8}=\frac{25}{8}$.

Now think of the shaded triangle QPD as having base PD. We thus need to find its height. It is easy to see that triangles QPD and QEA are similar. Thus, their bases are in the ratio $\frac{25}{8}$ to 5, which simplifies to $\frac{5}{8}$. So must be the ratio of their heights h_1 (of QPD) and h_2 (of QEA). But $h_1+h_2=5$. Given that $h_2=\frac{8}{5}h_1$ it follows that $h_1(1+\frac{8}{5})=h_1(\frac{13}{5})=5$. Simplify to $h_1=\frac{25}{13}$. So the height of ΔQPD is $\frac{25}{13}$. Therefore, our triangle has area $\frac{1}{2}\times\frac{25}{8}\times\frac{25}{13}$, which simplifies to $\frac{625}{208}$.

Problem 4:
The base of a right circular cylinder has diameter 6, and the height of the cylinder is 8. The cylinder is enclosed in a sphere which is just large enough to contain the cylinder. What common fraction of the volume of the sphere is taken up by the cylinder?
Solution:
Slice through the sphere, along the central axis of the cylinder. We get a cross-section that is a circle, with an inscribed rectangle with base 6 (the diameter of the cylinder) and height 8 (the height of the cylinder). By the Pythagorean theorem, the diagonal of the rectangle has length $\sqrt{6^2+8^2}$. This is the diameter of the circle, and hence of the sphere. So the sphere has diameter 10, and therefore radius 5.

The cylinder has base radius 3, and height 8, so its volume is $\pi\times3^2\times8$. The sphere has volume $\frac{4}{3}\times\pi\times5^3$. Divide $\pi\times3^2\times8$ by $\frac{4}{3}\times\pi\times5^3$ and simplify. The ratio of the volumes is $\frac{54}{125}$.

Problem 5:
An equilateral triangle with area 1 cm^2 is inscribed in a circle. What is the number of cm^2 in the area of the circle? Express the answer as a decimal, rounded to the nearest one-hundredth of a cm^2. Note that π is approximately 3.14159.

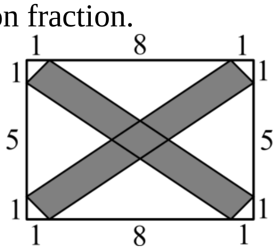


Solution:
Join the center of the circle to the 3 vertices of the triangle, and drop perpendiculars from the center to the 3 sides of the triangle. This divides the triangle into 6 congruent 30-60-90 triangles.

If r is the radius of the circle, then the 6 triangles each have hypotenuse r, and hence legs $\frac{r}{2}$ and $\frac{r\sqrt{3}}{2}$. Thus each triangle has area $\frac{1}{2}\times\frac{r}{2}\times\frac{r\sqrt{3}}{2}=\frac{r^2\sqrt{3}}{8}$, and therefore the full equilateral triangle has area $\frac{6\times r^2\sqrt{3}}{8}=\frac{r^2 3\sqrt{3}}{4}$. Since this area is 1, $r^2=\frac{4}{3\sqrt{3}}$.

The area of the circle is $\pi\times r^2$; so our desired area is exactly $\frac{4\pi}{3\sqrt{3}}$. Calculating to the nearest one-hundredth of a cm^2, we get 2.42.

Problem 6:
The rectangle below has length 10 cm and width 7 cm. An X-shaped figure (shaded) is drawn, with dimensions as shown. What is the area of the shaded figure, in cm^2? Express the answer as a common fraction.



Solution:
We will find the area of the non-shaded portion of the figure, and subtract this from the area of the rectangle.

There are 4 little triangles at the corners. Each has area $\frac{1}{2}\times 1\times 1$, for a total of 2 cm^2.

Now look at the large non-shaded triangle at the bottom. It has base 8. We will find its height. Drop a perpendicular from the apex of this triangle to the base. The triangle is divided into two equal parts. Look at the right-hand part. This is similar to an "easy to spot" right triangle in the picture that has legs 9 and 6. So if y is the height of our large non-shaded triangle, we have $\frac{y}{4}=\frac{6}{9}$.

It follows that $y=\frac{8}{3}$. So the large non-shaded triangle has area $\frac{1}{2}\times 8\times\frac{8}{3}=\frac{32}{3}$. The sum of the areas of the two large non-shaded triangles (top and bottom) is therefore $\frac{64}{3}$ cm^2.

Now we do the same thing with the non-shaded triangles at the left and right. Essentially the same idea shows that the sum of their areas is $\frac{75}{4}$ cm^2. Therefore, the sum of the areas of the non-shaded regions is $2+\frac{64}{3}+\frac{75}{4}$, which is $\frac{505}{12}$. The area of the shaded region is $70-\frac{505}{12}$, that is, $\frac{335}{12}$ cm^2.

2013 Vector Magazine Article A:

Problem 1:

The number 130 has 8 positive factors, namely 1, 2, 5, 10, 13, 26, 65, and 130. How many positive integers smaller than 130 also have 8 positive factors?

Solution:

Let p be a prime. Let $n = p^a$. The positive factors of n are p^0 (that is, 1), p^1, p^2, p^3, and so on up to p^a. So p^a has $a+1$ positive factors.

Let p and q be distinct primes, and let $n = p^a q^b$. Now, look at any d that is a factor of n. It, thus can be written as $d = p^i q^j$. The number of powers of p (that is, i) can be 0, 1, 2, $\ldots$, a; so the number of possible i's used to produce d is $a+1$. Similarly, the number of powers of q (that is, j) can be 0, 1, 2, $\ldots$, b; so the number of possible j's used to produce d is $b+1$. Thus, it is clear that the number of possible such d's is $(a+1)(b+1)$. So n has $(a+1)(b+1)$ positive factors.

Now, Let p, q, and r be distinct primes, and let $n = p^a q^b r^c$. Essentially the same argument as the one above shows that n has $(a+1)(b+1)(c+1)$ positive factors. This idea can be easily generalized to numbers n that have more than 3 distinct prime factors.

Now we look for numbers less than 130 that have exactly 8 positive factors. Such a number can have various shapes: (i) the number can be of the form p^7, where p is prime; (ii) the number can have shape $p^3 q^1$, where p and q are distinct primes; and (iii) the number can have shape $p^1 q^1 r^1$ where p, q, and r are distinct primes. We count the numbers less than 130 of each type. For type (i), there is only 1 small enough number, namely 2^7. For type (ii), there are several: $2^3 3^1$, $2^3 5^1$, $2^3 7^1$, $2^3 11^1$, $2^3 13^1$, and also $3^3 2^1$, for a total of 6. For type (iii), again listing in an organized order yields: $2^1 3^1 5^1$, $2^1 3^1 7^1$, $2^1 3^1 11^1$, $2^1 3^1 13^1$, $2^1 3^1 17^1$, $2^1 3^1 19^1$, $2^1 5^1 7^1$, $2^1 5^1 11^1$, and also $3^1 5^1 7^1$, for a total of 9. The sum of the 3 types gives the answer of 16.

Problem 2:

N is a two-digit positive number and M is a three-digit positive number. Given that N percent of M is 777, what is the value of N?

Solution:

This requires some detective work, and a bit of luck. 777 factors to $7 \times 3 \times 37$, so 777 is divisible by 37. Since $\frac{N}{100} = \frac{777}{M}$ it follows that $NM = 100 \times 777$. Thus one of N or M is divisible by 37. Also, M is less than 1000 so it is clear that N is at least 78, but less than 100. There is no multiple of 37 between 78 and 100. So, 37 must divide M. Note also that 100 must divide NM. Since N is greater than 75 it can contribute at most one factor of 5. Thus, we must get at least one factor of 5 from M. Now list the multiples of 37 greater than 777. We get 814, 851, 888, 925, 962, 999. Only 925 can contribute a 5. Thus, $M = 925.$ Check whether it works. If $\frac{N}{100} = \frac{777}{925}$, then $N = 84$. So our 2-digit integer is 84.

Problem 3:

How many ordered triples (a,b,c) are there such that a, b, and c are positive integers and $(a^b)^c = 64$? For example, one such triple is $(64,1,1)$.

Solution:

There are various systematic ways of counting. We describe one that is as efficient as most, and generalizes easily. We want $a^{bc} = 64$. So (i) $a = 64$, $bc = 1$, or (ii) $a = 8$, $bc = 2$, or (iii) $a = 4$, $bc = 3$, or (iv) $a = 2$, $bc = 6$. There is obviously 1 triple of type (i). For type (ii), we want to count the ordered pairs (b,c) with $bc = 2$. There are just as many of these as there are positive divisors of 2, so there are 2 triples of type (ii). The same argument shows that there are 2 triples of type (iii). To count the triples of type (iv), we need to count the ordered pairs (b,c) with $bc = 6$. There are just as many of these as there are positive divisors of 6, so there are 4 triples of type (iv). Adding up we get a total of 9 triples.

Problem 4:

Five students will work on problem-solving in groups. Any group can consist of 1 to 5 students and each student must belong to exactly one group. In how many ways can the 5 students be divided into groups?

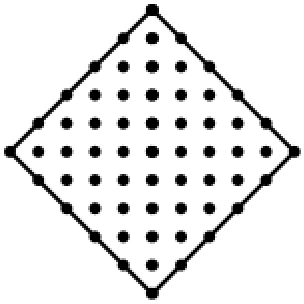
Solution:

There could be a single group consisting of all 5 students, or 5 groups of 1 student each, giving us 2 ways. Or else there could be a group of 4 and a group of 1 (we refer to this pattern as 4-1). So how many 4-1 groupings are there? The lone student can be chosen in 5 ways. Once this is done, everything is determined. So there are 5 groupings of type 4-1. Now we count the patterns of type 3-2. Choose the students who will be in the group of 2. This can be done in 10 ways. So there are 10 groupings of type 3-2. Next we count the groupings of type 3-1-1. The 2 lone students can be chosen in 10 ways, giving 10 groupings of type 3-1-1. Next we count groupings of type 2-1-1-1. The two paired students can be chosen in 10 ways, so there are 10 such groupings. We have left to the last the type 2-2-1, which is the trickiest to count. The lone student can be chosen in 5 ways. For every one of these ways, we are left with 4 students, who need to be broken up into 2 groups of 2. Think of the students as being P, Q, R, and S. Concentrate on P. The partner of P can be chosen in 3 ways. Once this is done, we have divided our 4 people into 2 groups. So there are 3 ways to divide 4 people into 2 groups of 2 (it is all too easy to mistakenly believe that there are 6 ways). So there are 5×3 groupings of type 2-2-1. Add up: $1+1+5+10+10+10+15=52$. We get a total of 52 ways to divide the 5 students into groups.

Problem 5:

How many ordered pairs (a,b) are there such that a and b are integers (not necessarily positive) and $|a|+|b|\le 5$?

Solution:



We want to count the number of dots in the above picture. The geometry suggests various ways of counting. For example, we could count the dots in each row, then add from top to bottom. We get $(1+3+5+7+9+11)+(9+7+5+3+1)$. We sum it up and get that the number of ordered pairs is 61. In the same way, we can find a general expression for the number of integer solutions of $|a|+|b|\le n$.

Problem 6:

What is the probability that the sum of two distinct randomly chosen positive factors of 420 is odd? Express the answer as a common fraction. Note that 1 and 420 are factors of 420.

Solution:

The positive factors of 420 can be counted by making an explicit organized list, but it is much easier to note that $420 = 2^2 3^1 5^1 7^1$. Thus, 420 has $(2+1)\times(1+1)\times(1+1)\times(1+1)$ positive factors (note the discussion of Problem 1). That is, 24 positive factors. The odd positive factors of 420 are the positive factors of $3^1 5^1 7^1$. Thus, 420 has 8 odd positive factors, and therefore 16 even positive factors.

Imagine that the two positive factors are chosen one after the other. Any choice can be thought of as an ordered pair (a,b). There are 24×23 such ordered pairs, all equally likely. Now we count the ordered pairs (a,b) that give an odd sum. These are of two types: (i) a odd, b even and (ii) a even, b odd. There are 8×16 ordered pairs of type (i), and 16×8 of type (ii), for a total of $2\times 8\times 16$. Our probability is therefore $\frac{2\times 8\times 16}{24\times 23}$, which simplifies to $\frac{32}{69}$.

Problem 7:

Boiling water, at 100 degrees Celsius, is to be cooled to below 5 degrees. The water cools by 10 degrees in the first minute (so it is 90 degrees at time $t=1$ minute). The water cools by 10×0.9 degrees in the second minute, by 10×0.9^2 degrees in the third minute, by 10×0.9^3 degrees in the fourth minute, and so on. At what **integer** number of minutes will the water temperature first be below 5 degrees? (Note: this problem requires calculator).

Solution:

After exactly 1 minute, (at $t=1$), the temperature is $100-10$. At $t=2$ it is $100-(10+10\times 0.9)$, at $t=3$ it is $100-(10+10\times 0.9+10\times 0.9^2)$, and so on. We can patiently compute. After (quite) a while, we find that the temperature first dips below 5 after an integer number of minutes when that integer is 29.

But it is useful here to use more machinery. Let $r = 0.9$. Then after n minutes the temperature is $100 - (10 + 10 \times r + 10 \times r^2 + \cdots + 10 \times r^{n-1})$. The geometric series $10 \times (1 + r + r^2 + \cdots + r^{n-1})$ has sum $10 \times \frac{1 - r^n}{1 - r}$. Let $r = 0.9$, and get:

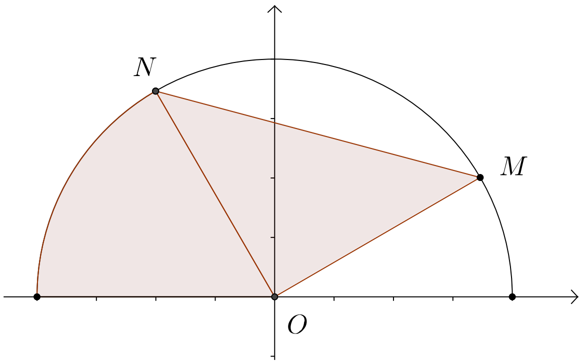
$$100 - 10 \times \frac{1 - 0.9^n}{1 - 0.9} = 100 - 10 \times \frac{1 - 0.9^n}{0.1} = 100 - 100 \times (1 - 0.9^n) = 100 \times (1 - (1 - 0.9^n)) = 100 \times 0.9^n .$$

So, after exactly n minutes the temperature is 100×0.9^n. We want this to be (barely) below 5, so we want 0.9^n to be (barely) below 0.05. Now experiment with a calculator. The smallest n such that $0.9^n < 0.05$ is $n = 29$.

2015 Vector Magazine Article A:

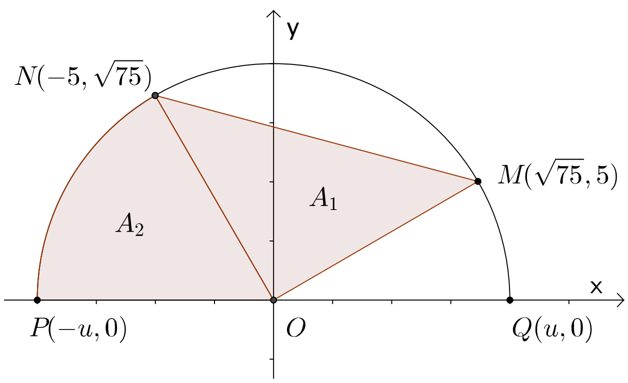
Problem 1:

A half-circle has centre at the origin $O(0,0)$. Points $M(\sqrt{75},5)$ and $N(-5,\sqrt{75})$ are on the half-circle as shown. What is the area of the shaded region, correct to the nearest integer?



Solution:

First, we redraw the figure, including the x and y axes, and the two ending points of the diameter, $P(-u,0)$ and $Q(u,0)$, of the circle on the x-axis as in the figure below.



Thus, the radius of the circle, u, is given by

(1) $u=\sqrt{\sqrt{75}^2+5^2}=\sqrt{75+25}=\sqrt{100}=10$.

Since the y-coordinate of M is 5, which is half of the radius of the circle, then $\angle MOQ=30°$. Using the same argument again we conclude that $\angle NOP=60°$. Therefore, $\angle MON=90°$ and ΔMON is a right triangle with sides 10.

The area, A_1, of ΔMON is

(2) $A_1=(u\times u)/2=(10\times 10)/2=100/2=50$.

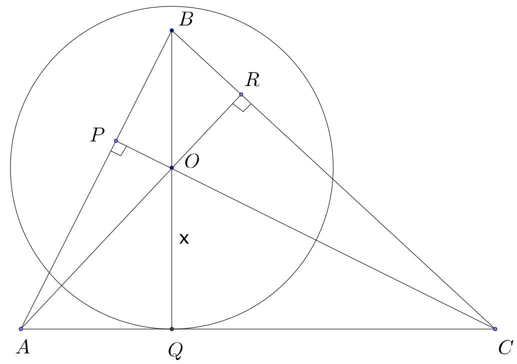
The area, A_2, of the $60°$ wedge NOP is

(3) $A_2=(\pi(10)^2)/6=100\pi/6\approx 314.16/6\approx 52.36\approx 52$ (rounded to the nearest integer).

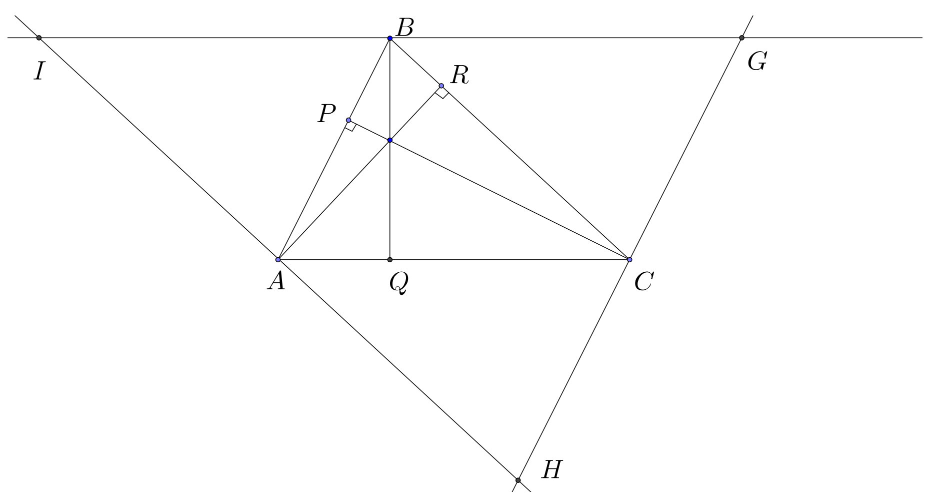
Thus, the area, A, of the shaded region is

(4) $A=A_1+A_2\approx 50+52=102$.

Problem 2:
Triangle ABC has $AB=10$ and $AC=14$. The three heights AR, BQ, and CP are drawn and meet at O. The distance AP is equal to 6. Let $OQ=x$, and draw the circle with centre O and radius x. What is the area of the circle? Express the answer as a decimal, correct to 1 decimal place.



Solution:
First, we have to prove that all three heights indeed meet at one point. To do this, we use the theorem, which is rather easy to prove, that all three perpendicular bisectors of any triangle meet at one point, which is also called the circumcenter of that triangle. We recommend that the reader may try to prove it. Now, in order to use this theorem, we redraw ΔABC, along with its three heights, along with a line through A parallel to BC, a line through B parallel to AC, and line through C parallel to AB. The three lines form another, larger, ΔGHI, as in the figure below.



Since $BCAI$ is a parallelogram, it turns out that $BC=AI$. For the same reason, $BCHA$ is a parallelogram and thus $BC=HA$. Thus, $HA=AI$ and, therefore, AR is a perpendicular bisector to side HI of ΔGHI. By the same argument, BQ and CP are perpendicular bisectors to sides GI and GH of ΔGHI. Since AR, BQ, and CP are perpendicular bisectors to sides of ΔGHI then they all meet at one point namely O.
Now that we know that the three heights meet at point O, we can solve the problem.
Since $AC=14$, $AP=6$, and ΔAPC is a right triangle, then

(1) $PC = \sqrt{AC^2 - AP^2} = \sqrt{14^2 - 6^2} = \sqrt{196 - 36} = \sqrt{160} = 4\sqrt{10}$.

It is easy to see that ΔAPC, ΔAQB, and ΔOQC are similar triangles. Since ΔAPC and ΔAQB are similar,

(2) $\dfrac{AQ}{AB} = \dfrac{AP}{AC} = \dfrac{6}{14} = \dfrac{3}{7}$. Thus,

(3) $AQ = \dfrac{3}{7} \times AB = \dfrac{3}{7} \times 10 = \dfrac{30}{7}$.

Since $AQ + QC = AC$,

(4) $QC = AC - AQ = 14 - \dfrac{30}{7} = \dfrac{98 - 30}{7} = \dfrac{68}{7}$.

Since ΔAPC and ΔOQC are similar,

(5) $\dfrac{OQ}{AP} = \dfrac{QC}{PC} = \dfrac{\frac{68}{7}}{4\sqrt{10}} = \dfrac{68}{28\sqrt{10}} = \dfrac{17}{7\sqrt{10}}$. Thus,

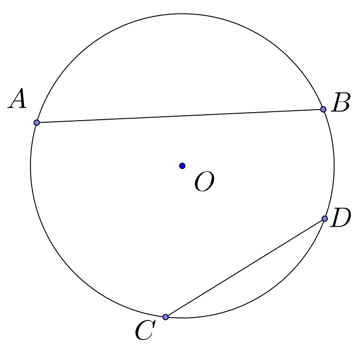
(6) $OQ = x = \dfrac{17}{7\sqrt{10}} \times AP = \dfrac{17}{7\sqrt{10}} \times 6 = \dfrac{102}{7\sqrt{10}}$.

Thus, W, the area of the circle with centre at O and radius x is

(7) $W = \pi x^2 = \pi(\dfrac{102}{7\sqrt{10}})^2 = \pi \dfrac{10404}{490} \approx 66.7$.

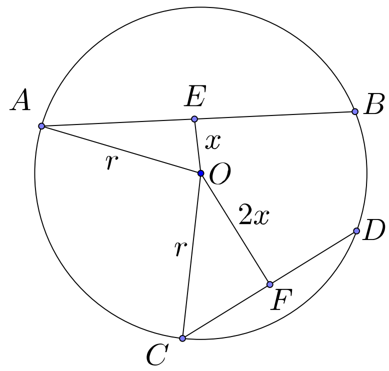
Problem 3:

In the circle below with centre O, chord AB has length 22, and chord CD has length 16. Chord CD is twice as far from the centre of the circle as chord AB. What is the square of the radius of the circle?



Solution:

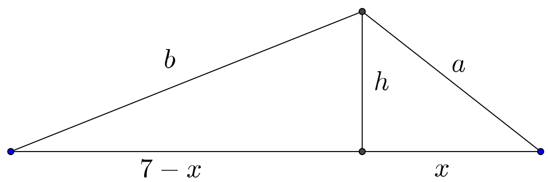
Redraw the figure, let E be the midpoint of AB, and let F be the midpoint of CD.



Thus, $AB \perp OE$, and ΔAEO is a right triangle. Let $OE = x$, and let $OA = r$.
(1) $AE = AB/2 = 22/2 = 11$. Thus,
(2) $x^2 + 11^2 = r^2$.
Similarly, let F be the midpoint of CD. Thus, $CD \perp OF$, and ΔCFO is a right triangle. Given that $OF = 2x$, given that $OC = r$, and given that
(3) $CF = CD/2 = 16/2 = 8$, it turns out that
(4) $(2x)^2 + 8^2 = r^2$.
We have two equations:
(5) $x^2 + 121 = r^2$, and
(6) $4x^2 + 64 = r^2$.
To solve for r^2 we multiply (5) by 4 to get:
(7) $4x^2 + 484 = 4r^2$.
Subtract (6) from (7) we get
(8) $4x^2 + 484 - (4x^2 + 64) = 4r^2 - r^2 = 3r^2$.
Switching sides and simplifying to get
(9) $3r^2 = 420$. Thus,
(10) $r^2 = \dfrac{420}{3} = 140$. So, the value of the square of the radius of the circle is **140**.

Problem 4:
A triangle has sides 3, 5, and 7. What is the square of its smallest height? Express the answer as a common fraction.
Solution:
First, we have to know which of the three heights is the smallest. For any triangle, its area, A, is given by
(1) $A = \dfrac{sh}{2}$, where s is any of the three sides, and h is the height to that side.
Therefore, the larger the side the smaller is the height to that side.
Suppose that the three sides of the triangle are $a = 3$, $b = 5$, and $c = 7$. From (1) we conclude that we need to find the height to side c.
Any triangle must be acute, right, or obtuse. If we draw the heights of an acute triangle, all three heights are drawn inside the triangle. In a right triangle, the two smaller sides are also heights while

the height to the hypotenuse is drawn inside the triangle. In an obtuse triangle the heights to the two smaller sides are drawn outside the triangle but the height to the largest side is drawn inside the triangle. Therefore, the height to side c is inside the triangle and it divides c into two segments, x and $7-x$.



Let h be the height to side c. Using the Pythagorean theorem we get

(2) $h^2+x^2=3^2$, and

(3) $h^2+(7-x)^2=5^2$.

From (3) we get that

(4) $h^2+49+x^2-14x=25$. Thus,

(5) $h^2+x^2=14x-24$.

Subtract (2) from (5) to get

(6) $0=14x-24-9$.

Therefore,

(7) $x=\frac{33}{14}$.

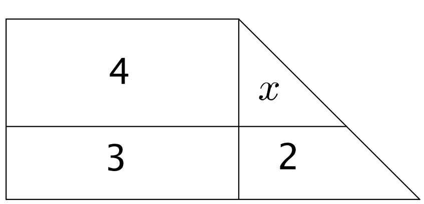
Therefore,

(8) $h^2=9-x^2=9-\frac{33^2}{14^2}=9-\frac{1089}{196}=\frac{1764-1089}{196}=\frac{675}{196}$.

So, the value of the square of the smallest height is $\frac{675}{196}$.

Problem 5:

In the picture below, lines that look perpendicular are perpendicular. The largest trapezoid of the picture is divided into a trapezoid, two rectangles, and a triangle as shown. The trapezoid has area 2, and the rectangles have area 3 and 4 as shown. What is the value of x, the area of the small triangle? Express the answer as a common fraction.



Solution:

The upper rectangle has area 4 and the lower rectangle has area 3. Both rectangles have the same base, therefore the ratio of their heights is $\frac{4}{3}$, and so is the ratio of the height of the small triangle to the height of the small trapezoid. Also, the ratio of the height of the small triangle to the height of the large triangle is $\frac{4}{7}$. Because the small triangle and the large triangle are similar triangles, the ratio of their bases is also $\frac{4}{7}$. Since the area of a triangle is (Base$\times$height)/2, it turns out the ratio of the area of the small triangle to area of the large triangle is

(1) $\frac{4}{7}\times\frac{4}{7}=\frac{16}{49}$.

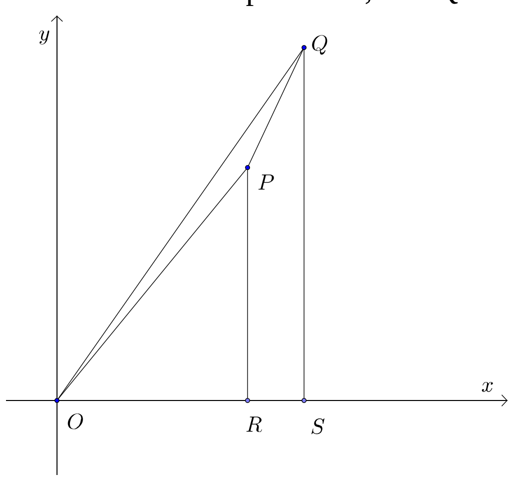
The small trapezoid is what is left of the large triangle after removing the small triangle. Thus, the ratio of the area of the small trapezoid to the large triangle is

(2) $\frac{49-16}{49}=\frac{33}{49}$.

Using a similar argument it turns out that the ratio of the area of the small triangle to the area of the small trapezoid is $\frac{16}{33}$. Given that the area of the small trapezoid is 2, it turns out that the area of the small triangle, x, satisfies

(3) $x=\frac{16}{33}\times 2=\frac{32}{33}$.

Problem 6:
What is the area of the triangle whose vertices have coordinates $(0,0)$, $(5,7)$, and $(7,10)$? Express the answer as a common fraction.
Solution:
Let $O(0,0)$, $P(5,7)$, and $Q(7,10)$ and draw ΔOPQ on the xy coordinate system and draw the vertical lines from points P, and Q to the x-axis (figure not drawn to scale).



The area, A_1, of ΔOQS is

(1) $A_1=(7\times 10)/2=70/2=35$.

The area, A_2, of ΔOPR is

(2) $A_2=(5\times 7)/2=35/2$.

The area, A_3 of trapezoid $RPQS$ is

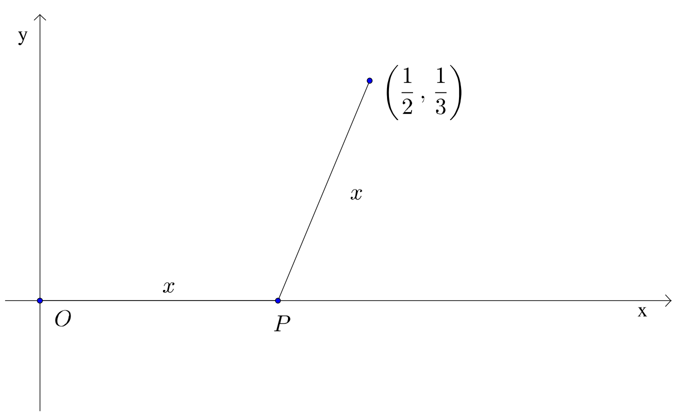
(3) $A_3 = ((10+7)\times 2)/2 = 17$.

Thus, the area, A, of ΔOPQ is

(4) $A = A_1 - (A_2 + A_3) = 35 - (35/2 + 17) = 35 - 69/2 = 70/2 - 69/2 = 1/2$.

Problem 7:
What is the area, in square metres, of the smallest square that both can be fully covered with no gaps or overlaps by using 50 cm by 50 cm tiles only, and also can be fully covered with no gaps or overlaps by using 40 cm by 60 cm tiles only?
Solution:
The total area in cm^2 is clearly an integer (since the areas of the tiles in cm^2 are integers). Since the measure of each side of the square (in cm) is an integer, and since we are looking for the area of the square, then the measure of the total area in cm^2 has to be a perfect square. It also needs to be a multiple of 2500 when we use $50cm \times 50cm$ tiles, and it needs to be a multiple of 2400 when we use $60cm \times 40cm$ tiles. Since $2500 = 2^2 5^4$, and $2400 = 2^5 3^1 5^2$, we conclude that the lowest perfect square that is a multiple of both numbers is $2^6 3^2 5^4$, or 360 000. So, the minimal area needed to be covered is 360 000 cm^2, or 36 m^2.

Problem 8:
What is the x-coordinate of the point P on the x-axis such that the distance from P to the origin is the same as the distance from P to the point with coordinates $(\frac{1}{2}, \frac{1}{3})$? Express the answer as a common fraction.
Solution:



Let $P(x,0)$ be a point on the x-axis. The distance of P from the origin is x, and the distance from P to $(\frac{1}{2}, \frac{1}{3})$ is $\sqrt{(x-\frac{1}{2})^2 + (\frac{1}{3})^2}$. If the distances are the same, so are their squares. Thus,

(1) $x^2 = (x-\frac{1}{2})^2 + (\frac{1}{3})^2 = x^2 - x + (\frac{1}{2})^2 + (\frac{1}{3})^2$.

Regrouping terms, multiplying by 36, and eliminating x^2 we get
(2) $36x = 9 + 4 = 13$.

Thus, $x = \frac{13}{36}$.

So, the x-coordinate of the point P is $\frac{13}{36}$.

2015 Vector Magazine Article B

Problem 1:

What is the smallest positive integer n such that $1+2+3+\cdots+(n-1)+n$ is a multiple of 100?

Solution:

First, let us find that for even n:

(1) $1+2+3+\cdots+(n-1)+n$

$$=(1+n)+(2+n-1)+\cdots+\left(\frac{n}{2}+\frac{n}{2}+1\right)=\frac{n(n+1)}{2}.$$

We leave to the reader to prove that the same works for odd n.

Thus, we need to find the smallest positive integer n such that $\frac{n(n+1)}{2}=100k$, where k is an integer.

(2) $n(n+1)=200k=2^3 5^2\times k$.

Given this, we know that either 5^2 divides n, or 5^2 divides $n+1$ - (but not both). So, either $n=m\times 25$, or $n+1=m\times 25$, where m is a positive integer.

Try the smallest possible case of $m=1$, or $n+1=25$.

(3) $n(n+1)=24\times 25=600=3\times 200$.

Thus, $n=24$.

Check: $1+2+3+\cdots+23+24=\frac{24\times 25}{2}=300$.

Problem 2:

Ten people, Alan, Beti, and 8 others, are divided at random into two groups, one group with 4 people, and the other group with 6 people. What is the probability that Alan and Beti end up in the same group? Express the answer as a common fraction.

Solution:

The probability that Alan is in the group of 4 is $\frac{4}{10}=\frac{2}{5}$. The probability that he is in the group of 6 is $\frac{6}{10}=\frac{3}{5}$. If he is in the group of 4, there is a need to include 3 more out of the remaining 9 people. Thus, the probability that one of them is Beti is $\frac{3}{9}=\frac{1}{3}$. Thus, the probability that both are in the group of 4 is $\frac{2}{5}\times\frac{1}{3}=\frac{2}{15}$.

Similarly, the probability that both are in the group of 6 is $\frac{3}{5}\times\frac{5}{9}=\frac{15}{45}=\frac{1}{3}$.

Thus, the total probability that they are in the same group is $\frac{2}{15}+\frac{1}{3}=\frac{2+5}{15}=\frac{7}{15}$.

Problem 3:
Note that $1+2+3+\cdots+7+8=36$, and 36 is a perfect square. What is the smallest perfect square greater than 36 which is the sum of the first n positive integers for some n?
Solution:
We use the identity in Problem 1. $n>8$ has to satisfy $\frac{n(n+1)}{2}=k^2$, where k is an integer. This is equivalent to requiring $n(n+1)=2k^2$. Since every integer has to be either even or odd, then, either (a) n is even in which case define $m=\frac{n}{2}$, or (b) $n+1$ is even in which case define $m=\frac{n+1}{2}$. Note that in both cases m is an integer. Since we are looking for a value $n>8$, we only need to check values of $m>4$.
In case (a), substituting $n=2m$: $2m(2m+1)=2k^2$ and, thus, $m(2m+1)=k^2$.
In case (b), substituting $n=2m-1$: $2m(2m-1)=2k^2$ and, thus, $m(2m-1)=k^2$.
Since m, $2m+1$, and $2m-1$ do not have the same factors (note that $2m+1$ and $2m-1$ are always odd), it follows that either (a) m, and $2m+1$ are perfect squares, or (b) m, and $2m-1$ are perfect squares.
In fact, we need to look for $m=h^2$, where either $2m-1=i^2$, or $2m+1=j^2$, where h, i, and j are relatively prime.
Let us look at the first few cases where $m>4$.
The smallest case where $m>4$ is a perfect square is when $m=3^2=9$.
In this case, neither $2m-1=17$, nor $2m+1=19$ are perfect squares. No good.
In the next case when $m=4^2=16$, $2m-1=31$, and $2m+1=33$. No good.
In the next case when $m=5^2=25$, $2m-1=49$, and $2m+1=51$.
49 is a perfect square, so we have a solution, namely: $m=25$, and $n=49$.
We check the solution:
$1+2+3+\cdots+48+49=49\times 25=1225=35^2$.
Thus, $n=49$.

Problem 4:
What is the smallest positive N such that $N\times 5!$ is a perfect cube?
Solution:
Let us rewrite $5!$ as powers of prime numbers:
$5!=1\times 2\times 3\times 4\times 5=2\times 3\times 2^2\times 5=2^3\times 3\times 5$.
Since $N\times 5!$ needs to be a perfect cube, $N=3^2\times 5^2=9\times 25=225$.
Check the solution: $N\times 5!=225\times 120=27000=30^3$.

Problem 5:
You toss 2 dice and record the sum. Then you do it again. What is the probability that the recorded sums are the same? Express the answer as a common fraction.
Solution:
There are 11 possible different sums of 2 dice. For example: the probability to get sum of 2 is $\frac{1}{36}$ and is the same as the probability to get sum of 12. The probabilities to get each of the 11 different sums, from 2 to 12, inclusive, are: $\left\{\frac{1}{36}, \frac{1}{18}, \frac{1}{12}, \frac{1}{9}, \frac{5}{36}, \frac{1}{6}, \frac{5}{36}, \frac{1}{9}, \frac{1}{12}, \frac{1}{18}, \frac{1}{36}\right\}$.
For the two consecutive tossed sums to be identical, we have to sum the squares of each of the above 11 probabilities. Let the sum be p:
$$p = 2 \times (\frac{1}{36})^2 + 2 \times (\frac{1}{18})^2 + 2 \times (\frac{1}{12})^2 + 2 \times (\frac{1}{9})^2 + 2 \times (\frac{5}{36})^2 + (\frac{1}{6})^2.$$
Thus, $p = \frac{2\times(1+4+9+16+25)+36}{36\times36} = \frac{2\times55+36}{1296} = \frac{146}{1296} = \frac{73}{648}$.

Problem 6:
You can use three different taps, alone or in combination, to fill a pool. If you use taps B and C only, it will take 9 hours to fill the pool. If you use all three taps (A, B, and C), it takes 7 hours. Tap B can fill the pool on its own in half the time it takes tap A on its own. How many hours would it take for tap C to fill the pool on its own?
Solution:
Let a, b, and c be the number of hours it takes to fill the pools using, respectively, taps A, B, and C alone. Then, the flow rates for taps A, B, and C are $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$, respectively, where the units of these rates is pools per hour. Since it takes 9 hours if using taps B and C only, then:
(1) $\frac{1}{b} + \frac{1}{c} = \frac{1}{9}$.
Similarly,
(2) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{7}$.
Thus, subtracting (1) from (2):
(3) $\frac{1}{a} = \frac{1}{7} - \frac{1}{9} = \frac{9-7}{63} = \frac{2}{63}$.
Thus, $a = \frac{63}{2}$, and $b = \frac{a}{2} = \frac{63}{4}$, and $\frac{1}{b} = \frac{4}{63}$.
Thus, from (1):
(4) $\frac{1}{c} = \frac{1}{9} - \frac{4}{63} = \frac{7-4}{63} = \frac{3}{63} = \frac{1}{21}$.
Thus, $c = 21$, so it takes 21 hours for tap C to fill the pool on its own.

Problem 7:
Betty and Ben each select independently and at random an integer between 0 and 5 (inclusive). Call these two numbers m and n. What is the average value of $|m-n|$? Express the answer as a common fraction.
Solution:
Each selects at random out of 6 different integers. If Betty selects the number 0 (probability of $\frac{1}{6}$), the average value of $|m-n|$ is $\frac{0+1+2+3+4+5}{6}=\frac{15}{6}$.
The same average value occurs if Betty selects the number 5.
If Betty selects the number 1, the average value is $\frac{1+0+1+2+3+4}{6}=\frac{11}{6}$.
This is the same average if Betty selects the number 4.
If Betty selects the number 2, the average value is $\frac{2+1+0+1+2+3}{6}=\frac{9}{6}$.
Again, the same average occurs if Betty selects the number 3.
Thus the average over all possible choices, (since they are all equally likely), is:
$\frac{2(\frac{15}{6}+\frac{11}{6}+\frac{9}{6})}{6}=\frac{15+11+9}{6\times 3}=\frac{35}{18}$.

Problem 8:
What is the greatest integer n for which $\frac{24n}{n-4}$ is an integer?
Solution:
Note that $\frac{24n}{n-4}=\frac{24(n-4+4)}{n-4}=\frac{24(n-4)}{n-4}+\frac{24\times 4}{n-4}=24+\frac{96}{n-4}$.
Thus, the largest n such that $\frac{96}{n-4}$ is an integer is $n=100$.
Let us check the solution. When $n=100$: $\frac{24n}{n-4}=\frac{24\times 100}{100-4}=\frac{2400}{96}=25$ (an integer!).