Combinatorics: Overcount

Theory: Sometimes, it is easy to count by overcounting on purpose, then divide!

Example: If 5 people are to sit at a circular table, how many ways can you arrange the seating (Note: rotation of an arrangement is considered the same.)

Solution: Ignore the rotation restriction, we have 5! ways. Now, bring back the restriction. Since there are 5 people, there are exactly 5 ways to rotate each seating (ABCDE, BCDEA, CDEAB, DEABC, EABCD). Thus, just divide by 5 and we get the value to be $\frac{5!}{5} = 24$.

Excercise: What about for 6 people? 7 people? N people? What if you say that reflection is also considered the same arrangment?

Example: How many interior diagonals are in a hexagon?

There are 6 vertex. It can form a interior diagonal with vertex that is not adjacent or itself. Thus, it can form with $6 - 3$ vertex. Hence, we have $6 \times 3 = 18$. However, segment AB and BA are the same, so we divide by 2 to get 9 $\square$

Example: (AIME) A convex polyhedron has for its faces 12 squares, 8 regular hexagons, and 6 regular octagons. At each vertex of the polyhedron one square, one hexagon, and one octagon meet. How many segments joining vertices of the polyhedron lie in the interior of the polyhedron rather than along an edge or a face?

We apply the same principle! First, count the number of vertex. It is equal to $12 \times 4 = 8 \times 6 = 6 \times 8 = 48$. For each of the 48 vertex, let's see how many vertex they can connect to such that it forms a interior diagonal. Each vertex is part of a square, hexagon and octagon at the same time. There are $4 + 6 + 8$ vertex that goes along an edge (or the vertex itself). But we are overcounting few cases (a vertex can be at square and hexagon at the same time, etc.) Indeed, we are overcounting 4 cases (a vertex at square and hexagon at the same time, square and octagon at the same time, hexagon and octagon at the same time and square, hexagon and octagon at the same time [we actually counted this 2 times).

Thus, we **have** $48 - 18 + 4 + 1 = 35$ vertex that form a segment interior of the polyhedron for each of the 48 vertex.

Then we have 48×35. Unfortunately, you are not done. We are overcounting something. Indeed, segment AB and BA are the same thing! We must divide by 2 and we get $\frac{48 \times 35}{2}$ and have $\boxed{840}$.

Now apply these theories to problems!

1. An international meeting is held between England, Germany, and France. Three representatives attend from England, four from Germany, and two from France. How many ways can all nine representatives sit around a circular table, if representatives of the same country sit together? (Two ways are considered the same if one can be rotated to produce the other.)

2. In how many ways can 7 people sit around a round table if no two of the 3 people Pierre, Rosa, and Thomas can sit next to each other?

3. Given 5 colors to choose from, how many ways can we color the four unit squares of a 2×2 board, given that two colorings are considered the same if one is a rotation of the other?