Math Challengers 2017 Solutions

Bull's Eye and Co-op rounds

Math Challengers 2017 Provincial Bull's-Eye Solution

1. The travel distances, in km, between point A and point B are 60 by air, 91 by land, and 121 sea. The travel distances between point B and point C are (respectively): $75, 122$ and 75. The travel distances between point C and point A are (respectively): $90, 100$ and 105. Greg starts at A, travels to B, then to C, then back to A. What is the shortest distance he culd travel in km, if he must use all three modes of transportation exactly once each.

Going from A to B is shortest compared to travelling to other location when travelling by air and land. Going from B to C is shortest compared to travelling to other location when travelling by sea. Hence, it is clear that one should choose to travel by sea when goint from B to C.

Now we have two choices choose A to B from air or choose A to B from land. We get the total distances to be 235 and 256 respectively. Therefore, the minimum distance is $\boxed{235}$.

2. On TV, between 12 Midnight and 6 AM there are 5 minutes of advertisement during each hour. Between 6 AM and 11 AM there are 15 minutes of advertisement during each hour. Between 11 AM and 4 PM there are 10 minutes of advertisement during each hour. Between 4 PM and 12 Midnight there are 20 minutes of advertisement during each hour. What is the average daily advertisement rate on TV, in percent, rounded to the nearest integer, throughout the entire period of 24 hours?

From 12 Midnight to 6 AM, there are 5 minutes of advertisement each hour which means $\frac{1}{12}$ of the show is advertisement for 6 hours.

From 6 AM to 11 AM, there are 15 minutes of advertisement each hour which means $\frac{1}{4}$ of the show is advertisement for 5 hours.

From 11 AM to 4 PM, there are 10 minutes of advertisement each hour which means $\frac{1}{6}$ of the show is advertisement for 5 hours.

From 4PM to 12 Midnight there are 20 minutes of advertisemtn each hour which means $\frac{1}{3}$ of the show is advertisement for 8 hours.

Therefore we have the total percentage to be $\frac{6\times\frac{1}{12}+5\times\frac{1}{4}+5\times\frac{1}{6}+8\times\frac{1}{3}}{24}=\frac{3}{24}=\frac{1}{8}=0.125\approx\boxed{13}\%$

3. Two painters, A and B, are painting a room. They paint at the same constant rate, but A started painting later than B did. An hour ago, A has only painted $\frac{1}{4}$ as large an area as B had. By now, A has painted a total area $\frac{1}{2}$ as large as B had. How many minutes from now will A have painted a total area $\frac{3}{4}$ as large as B will have painted by then?

Suppose A and B paint $\frac{1}{y}$ of the room in one hour and B originally painted $4x$ and A painted x.

We have $4x+\frac{1}{y}=2\times(x+\frac{1}{y})$

$2x=\frac{1}{y}$.

Let the answer to our problem be k hours (we will convert to minute later). Then we have
$4x+\frac{1}{y}+\frac{k}{y}=\frac{4}{3}\times(x+\frac{1}{y}+\frac{k}{y})$.

Plugging in $\frac{1}{y}=2x$, we get:

$6x + 2kx = \dfrac{(12+8k)x}{3}$

$18x + 6kx = 12x + 8kx$

$6x = 2kx$. $k = 3$. Converting this to minute, we get $\boxed{180}$.

4. Sam planned to drive from point A to point B at speed of $90\dfrac{km}{h}$ (kilometres per hour). At point C on the way, $60km$ away from point B, the car broke down and a tow truck was called immediately, and left 42 minutes later from point B, at speed of $120\dfrac{km}{hr}$, reached point C, and immediately towed Sam and the stalled car to point B, at speed of $75\dfrac{km}{hr}$. It took Sam twice as much time as originally planned to get to point B from point A. Note that the diagram is not drawn to scale. What is the distance in km from point A to point B.

Let the distance from point A to B be x. Using $distance = time \times rate$, we get the planned time to be $\dfrac{x}{90}$.

From point A to C it took $\dfrac{x-60}{90}$ hours.

Waiting took $\dfrac{42}{60} = \dfrac{7}{10}$ hours.

The trick took $\dfrac{60}{120} = \dfrac{1}{2}$ hours.

The towing took $\dfrac{60}{75} = \dfrac{4}{5}$ hours.

Adding them up, we have:

$\dfrac{x-60+63+45+72}{90}$ hours which also equals to $\dfrac{2x}{90}$ hours as the problem stated.

Thus, we have $x + 120 = 2x$. $x = \boxed{120}$

5 What is the remainder when $2001 + 2002 + 2003 + ... + 2015 + 2016$ is divided by 2017?

Take mod! Mod means remainder, for example, $15 \equiv 3 \mod 12$ since when dividec by 12, 15 has a remainder of 3.

You can add or subtract by the number youre are dividing by and it will still maintain its original remainder! For example $(3+5) \equiv 8 \equiv 3 \mod 5$ (Do you see why?)

Here is a neat trick, subtract all the numbers we are adding by 2017. For example, we have:

$2016 \equiv -1 \mod 2017, 2015 \equiv -2 \mod 2017$. Our equation becomes much simpler and it reduces to $-(1+2+..+16) \mod 2017$.

$-(136) \mod 2017$. Now add 2017 and we get: $2017 - 136 = \boxed{1881}$.

6. A group of student has 12 members, 4 from Burnaby, 4 from Richmond, and 4 from Vancouver, Three students of the group are chosen at random to form a committee (of 3 students). What is the probability that the committees has a member from each of the 3 cities? Express the answer as fraction in lowest terms.

In total, we have $\dbinom{12}{3}$ combination.

We have 4 people to choose from Burnaby, Richmond and Vancouver respectively. Thus, we have $4^3 = 64$ ways.

Thus, our probability is $\dfrac{64}{\dbinom{12}{3}} = \boxed{\dfrac{16}{55}}$.

7 How many integers from 1 to 20 (inclusive) can be written as the difference of the squares of two integers? Note, for example that $1 = 1^2 - 0^2$, so 1 can be written as the difference of the squares of two integers.

When dealing with perfect squres, it is often a good technique to look at something $\mod 4$ (Sometimes $\mod 3$ works too but not this case). Perfect square can only be 1 or $0 \mod 4$ (prove this!). Thus, our possible differences are $(0, 1, 3)$ which means any number that is even but not divisible by 4 cannot be expressed as difference of two squres. That eliminates $\frac{1}{4}$ of the 20 integers.

All odd numbers can be expressed as differences of two squares. Indeed, take the difference between two consecutive squres! $1^2 - 0^2 = 1, 2^2 - 1^2 = 3, 3^2 - 2^2 = 5$. It seems that it generates every odd number! Proof of this is quite simple, $(n + 1)^2 - n^2 = 2n + 1$. n can be any nonnegative integer; thus, we can generate all odd numbers.

All numbers divisible by 4 can be expressed as differences of two squres. In fact, take the difference between the squres of two numbers that differ by 2. $2^2 - 0^2 = 4, 3^2 - 1^2 = 8, 4^2 - 2^2 = 12$. It seems that it generates every multiples of four! Proof of this is also quite simple, $(n + 2)^2 - n^2 = 4n + 4$. n can be any nonnegative integer; thus, we can generate all multiples of 4.

Thus, the only numbers that cannot be expressed as differences of two squres are even number not divisible by 4. That is $\frac{3}{4}$ of 20 (WARNING: if 20 is not divisible by 4, you cannot simple multiply by $\frac{3}{4}$.) $= \boxed{12}$

8. What is the value of 0.03170170170... + 0.53163163163...? Provide your answer as fraction in lowest terms.

In general, any decimal in the form $0.abcabcabc...$, where a, b, c are digits, can be represented in the form $\frac{abc}{999}$.

To prove this, let $0.abcabcabc... = x$. We have $abc.abcabcabc... = 1000x$.

$abc + x = 1000x$, $abc = 999x$, $\frac{abc}{999} = x$. (This can work for any repeated decimal!).

Now, let us apply this fact to this problem!

$0.03170170170... = \frac{3}{100} + \frac{170}{99900}$.

$0.53163163163... = \frac{1}{2} + \frac{316}{9990}$.

Add and we get $\frac{1}{2} + \frac{3}{100} + \frac{333}{9990} = \frac{150 + 9 + 10}{300} = \boxed{\frac{169}{300}}$.

9. The figure below is made of 10 identical squres. If the perimeter of the figure is 1 metre, (m), what is the area, in square metres, (m^2), of the figure. Express the answer as common fraction in lowest terms.

Unfortunately no diagram in this solution.

Let the side length of each squres be x. Then the perimeter is $16x = 1$. $x = \frac{1}{16}$. The area of each square is $\frac{1}{256}$, there are ten squres; thus the area of the entire figure is $\frac{10}{256} = \boxed{\frac{5}{128}}$

10. You combine 3 identical triangles with edges $3, 4$ and 5 to form a convex pentagon. What is the largest possible perimeters of this pentagon?

Our total angle of the triangles are $180 \times 3 = 540$. The sum of the interior angles of the pentagon is 540. To make a convex pentagon, all of the angles of the triangles must be used.

For this condition to happen the pentagon must be two parallelgram merged with one of the right triangles being the overlap. If this is to happen, the perimeter is either $(5 + 3 \times 3 + 4)$ or $(5 + 3 \times 4 + 3)$. We take the largest one which is $\boxed{20}$.

Note: This is pooly explained solution to the problem and I am not sure if I am right, I will try to think of something better.

11. For shipping a box, Canada Post charges either by its dimentions (\$40.00×(the total measures in metres (m) of its length + width + height)), or by its weight \$4.00× number of kg) - whichever of the two ways of calculation that is larger. A box carrying the Math Challengers trophies is shipped. Its dimentions are 40 by 60 by 80 centimetres, (cm), and its average density is $0.1\frac{kg}{litre}$. Note that one $litre$ is $1000cm^3$. What was the shipping cost, in dollars, correct to 2 decimal places?

We calculate each possible price.

Price by its dimentions:

$(\$40.00 \times (0.4 + 0.6 + 0.8)) = \$72.00.$

Price by its weight:

$\$4.00 \times (40 \times 60 \times 80)cm^3 \times \frac{1litre}{1000cm^3} \times \frac{0.1kg}{1litre}) = \$76.80.$

Clearly price by its weight is more expensive. $\boxed{76.80}$.

12. A tetrahedron is a body consisting of 4 vertices and 4 idential triangular faces as in the schematic figure below. If the height of the tetrahedron is of length 1, what is the toal surface area of thetetrahedron? Express your answer as $\frac{L\sqrt{M}}{N}$ where L, M, and N are prime.

Errata: This problem has a major flaw in that it assums that all tetrahedron with 4 congruent faces must have all the faces to be congruent. Indeed this is not true meaning that it does not have a unique height. The problem itself assumes that the tetrahedron is *regular*, which refers to a tetrahedron with all of its faces being equilateral.

Imagine (or better, DRAW!) the tetrahedron $ABCD$. Draw the height from A to the triangle BCD. Let us call the point of intersection, E. Let's call our edge length, a. We could go through very long ugly algebra to solve for a. However, let's think about it. Clearly $BE = CE = DE$ (since ABE, ACE and ADE are congruent). Thus, BE is the circumradius of the equilateral triangle BCD! Lets draw a perpendicular from E to BC and call the intersection F . Since this is the circumradius, it is clear that $\angle EBF = 30$. We have a $30 - 60 - 90$ triangle!. BF is half the length of BC so it equals to $\frac{a}{2}$.

Thus, we obtain $BE = \frac{a}{\sqrt{3}}$.

Triangle AEB is right at E. By pythagoras,

$AE^2 + BE^2 = AB^2$. $1 + \frac{a^2}{3} = a^2$.

$a^2 = \frac{3}{2}$.

Area of an equilateral triangle with side length s is $\frac{s^2\sqrt{3}}{4}$. (To prove this, just use $30 - 60 - 90$ triangles (cut the triangle in half!)). In this case, there are four of those triangles and we know the value of a^2.

Thus, we have $\frac{a^2\sqrt{3}}{4} \times 4$. $a^2 = \frac{3}{2}$.

Therefore, $\boxed{\frac{3\sqrt{3}}{2}}$

Here is a quick solution!

A formula for the height of a regular tetrahedron with edge length, a, is $\frac{\sqrt{6}a}{3}$. Since height is 1, we get the edge length to be $\frac{3}{\sqrt{6}}$.

Formula for the area of an equilateral triangle with side length s is $\frac{\sqrt{3}a^2}{4}$. Our tetrahedron has 4 of them; hence, the surface area must be $\sqrt{3}a^2$. We have a to be $\frac{3}{\sqrt{6}}$. Plugging it in, we get $\boxed{\frac{3\sqrt{3}}{2}}$.

Math Challengers 2017 Provincial Coop Solution

1. If you are allowed to rearrange the order of the digits in the number 20170408, how many different numbers can you get? Note that all digits of the original number must be used and the digit 0 cannot be the first digit.

This is constructive counting except we have a restriction. In problems like these, deal with the restriction first! 0 cannot be in the first digit is what restrict the counting so let us place the 0 first. 0 cannot be in the first digit, so it can be in the rest of 7 digits. Thus, we can choose 3 of the 7 digits to be 0 which can be done in $\binom{7}{3}$ ways.

Now, deal with the rest of the 5 DIFFERENT numbers. There are 5 slots left, so there are 5! ways.

Now multiply: $5! \times \binom{7}{3} = \boxed{4200}$

2. Let $F(1,1) = 1$. For $n > 1, F(n,1) = 4nF(1,1)$. For $m > 1, F(n,m) = 5mF(m,1)$. What is the value of $F(3,6)$.

$F(3,6) = 30F(3,1)$

$(3,1) = 12F(1,1)$

$F(3,6) = 30 \times 12 \times F(1,1) = \boxed{360}$

3. An integer consist of 30 digits such that any 7 consecutive digits are different. What is the maximum possible sum of all the 30 digits.

To do this, use the largest possible number for a digit. In this case we need 7 consecutive numbers so 9 to 3. We just loop 9...3 for 4 times and we have 28 digits, now just add 98. Which then will become $(9 + 8 + ... + 3) \times 4 + (9 + 8) = \boxed{185}$.

4. As in Question 3, an integer consists of 30 digits such that any 7 consecutive digits are different, and also such that each digit appears at least once. What is the minimum possible sum of all the 30 digits if the digit 9 appears at least 3 times?

We want to use as many 0s as possible. We can't let it be the first digit (obviously) so make the it the secondmost left digit and fill the every seventh digit with 0. Repeat the same process with 1 except make it the leftmost digit. Repeat the same process with $2, 3, 4$. Now, we filled 22 numbers. We need to fill in $5, 6, 7, 8, 9,$. We must fill 9 for 3 times, we there are 5 digits left to fill. Use 5 for 2 times and use the other number 1 time each.

Now we have, $0 \times 5 + 1 \times 5 + 2 \times 4 + 3 \times 4 + 4 \times 4 + t \times 2 + 6 + 7 + 8 + 9 \times 3 = \boxed{99}$ (I know this solutions is really hard to follow, so draw 30 digits and do the process and it won't be so difficult!)

5 On the desk, there are 11 identital \$1 coins. In how many ways you can divide them into one or more pils of ocins so that each pile has at least 2 coins?

We approach this by casework by number of piles:

Not much work can be shown. Tips to this is to be as ORGANIZED AS POSSIBLE! Notice the organization of each case.

Case 1: only 1 pile.

This one is simple, just a pile of 11 is the only solution. Thus, we have 1.

Case 2: Only 2 piles.

We have $2 + 9$, $3 + 8$, $4 + 7$, $5 + 6$.

4 solutions.

Case 3: Only 3 piles.

We have $(2+2+7)$, $(2+3+6)$, $(2+4+5)$

$(3+3+5)$, $(3+4+4)$

5 solutions.

Case 4: Only 4 piles.

We have $(2+2+2+5),(2+2+3+4),(2+3+3+3)$.

3 solutions.

Case 5: Only 5 piles.

We have $(2+2+2+2+3)$

1 solution.

We cannot have more piles since for 6 piles, the minimum sum is $2 \times 6 = 12 > 11$.

Thus, we just add all the values we counted!

$1+3+5+4+1 = \boxed{14}$.

6. Let $5^{2a} + 5^{3b} + 5^{4c} + 5^{5d} + 5^{6e} = 5^x$, where a, b, c, d, e, and x are positive integers. What is the smallest possible value of x.

Informal solution:

One easy way is to realize that if $5^{2a} = 5^{3b} = 5^{4c} = 5^{5d} = 5^{6e} = 5^y$ for some integer y, then the equation becomes $5^y \times 5 = 5^x$ which is always true!

We can see that the smallest possible value is when we take the lcm of $2, 3, 4, 5, 6$ which equals to 60. Thus we have $5 \times 5^{60} = 5^x = 5^{61}$. Therefore, $\boxed{61}$

Formal solution:

To show that why the above must be true, take this number base 5. We have $1000...0_5 + 1000...0_5 + 1000...0_5 + 1000...0_5 + 1000...0_5 = 1000...00_5$. Cearly, all of the number in the left hand side must be equal. Thus take the lcm like above and we get it to be $\boxed{61}$

7 The surface area of the cone below is $1000cm^2$. The ratio of the height of the cone to the radius of its base is $\frac{3}{2}$. What is the volume of the cone, rounded to the nearest $100cm^3$? An answer of 1100 is of the correct format while an answer of 1076 is of the wrong format.

The surface area of a cone is $\pi \times r \times l + \pi r^2$ where l is the slanted height. (To derieve this, try to think of the none circular part of the cone to be part of a sector with circumference $2\pi l$ with the ark length of $2\pi r$.

To get l, let us use pythagoras. $r^2 + (\frac{3r}{2})^2 = l^2$. $l = r\frac{\sqrt{13}}{2}$

Apply our formula and we have our surface area to be $1000 = \pi r^2 + \pi r^2 \frac{\sqrt{13}}{2}$

We get $r = \sqrt{\dfrac{1000}{\pi(1 + \dfrac{\sqrt{13}}{2})}}$

To find the volume, apply $pi \times r^2 \times h \times \frac{1}{3}$.

$h = \frac{3r}{2}$. We know the value of r. We have a calculator. Plug this value in. I obtained $1901.1332922655583914682293509899 \approx \boxed{1900}$

8. What is the remainder if you divide 2^{31} by 31?

This is an extremely challenging problem if one does not know Fermat's Last Theorem (Or Euler's Theorem) but becomes trivial if one does know it.

From what I see, if I do not use Fermat's Last Theorem, I would just be computing $2^{31} \mod 31$ with hand and that is not pretty to look at!

So, let me state what the theorem and prove it. This proof will be quite advanced! If you do not understand it, do not worry about it!

Fermat's last theorem: If p is a prime and a is an integer, then $a^p \equiv a \mod p$ (Notice how the problem becomes trivial if someone knows this theorem).

$\mod$ means remainder. $1 \mod 2$ means has a remainder of 1 if divided by 2.

Clearly, when $a = 0$, this fact is true. Now we are going to show that if it works for a, it will also work for $(a+1)^p$. In other words, $(a+1)^p \equiv a + 1 \mod p$.

$(a+1)^p - (a+1) = a^p + \binom{p}{1}a^{p-1} + ... + \binom{p}{p-1}a + 1 - (a+1)$.

By the binomial theorem. (If you do not know this, look it up and try to prove it on your own!)

This simplifies to $(a^p - a) + \binom{p}{1}a^{p-1} + \binom{p}{2}a^{p-2} ... + \binom{p}{p-1}a$. Because p is prime, we know that p divides $\binom{p}{k}$ for $1 \leq k \leq p - 1$. (Expand the combination!).

Hence, $\binom{p}{1}a^{p-1} + \binom{p}{2}a^{p-2} ... + \binom{p}{p-1}a \equiv 0 \mod p$ and from our hypothesis $(a^p - a) \equiv 0 \mod p$. $0 + 0 \equiv 0 \mod p$. Since it works for 0, it works for $0 + 1 = 1$. Since it works for 1, it works for $1 + 1 = 2$ and it continues! Therefore we proved it! (If you understood all the way to here, congratulation! Your level of math is extremely solid!).

So we use this theorem and we get $\boxed{2}$.

9. You want to construct a $400m$ (metres) running track consisting of 2 straight segments of $100m$ each, and 2 half circles with the same radius, r. What is ther adius r, in m, of the half circles? Note that the figure is not drawn to scale. Express your answer correct to 1 decimal place.

Unfortunately, we do not have a diagram, so please draw your own diagram.

The 2 circumferences of the half circles must add up to 200. Thus, we have $2\pi r = 200$.
$r = \frac{100}{\pi} = 31.8309886183790672 \approx \boxed{31.8}$

10. In Question 9, the track consists of 9 paralle lanes, and each lane of the track is $1.25m$ wide. If the $400m$ is measured as the length of the inner boundary of the inner most lane, what is the total length, in m, of the outer

boundary of the outer most lane, round to the nearest integer.

The radius gets bigger by $1.25m$ for each outer lane. We are at the ninth outer lane so we have $r+9*1.25=r+3.25$. We know $r=\dfrac{100}{\pi}$.

The straight line does not change its length. So we have $200+2\pi*(\dfrac{100}{\pi}+2)=470.6858347057703481558\approx\boxed{471}$

11. You have 51 cards numbered $1,2,3,...,50,51$ (a different number for each card). You remove one card at random , look at it, and notice that it is divisible by 5. You then, remove a second (different) card at random. What is the probability that the number of the second card is divisible by 3? Express your answer as fraction in lowest term.

We have 2 cases.

Case 1: The first card thrown was divisible by 3.

Only 3 numbers satisfy this property $(15,30,45)$.

Now for the number divisible by 3, there are $(\dfrac{51}{3}-1)=16$ cards divisible by 3 left. Thus, case 1 can be done in 3×16 ways.

Case 2: The first card thrown was not divisible by 3.

In total there are 10 cards divisible by 5. $(5,10,...,50)$. Out of these 3 are divisible by 3 as well. So there are 7 numbers.

Now for the second number, there are $\dfrac{51}{3}=17$ cards divisible by 3. Thus, case 2 can be done in 7×17 ways.

This is conditional probability! The total case is not everything, instead, it is all the cases where the first number was divisible by 5.

This can be done in 10×50 ways (10 from the first number divisble by 5 and 50 from the fact that there are $(51-1)$ cards left on the second time one throws the card).

Now we divide and we have $\dfrac{3\times 16+7\times 17}{10\times 50}=\boxed{\dfrac{167}{500}}$.

12. How many 5−digits numbers are there, using the digits $1,2,3,4,$ and 5, exactly once each, such that at most one digit is larger than its preceding digit? Note: 54321 and 52431 are allowed, while 52341 is not allowed.

We are going to use bunch of caseworks!

We can use casework based on the position of 5s

5 as the unit digit.

5 will be the largest, so it will definitely be bigger than its preceding digit. Thus, the other 4 digits must be consecutive.

1 way.

5 as the tens digit.

We have 4 choices for the unit digit. Using the same argument above, we know that the non chosen 3 digits must be consecutive.

4 ways.

5 as the hundreds digit.

We have $\binom{4}{2}$ choices to be the unit and tens digit. It is not 4×3 since only one of them will have the preceding digit to be larger. Use same argument from above.

$\binom{4}{2} = 6$ ways.

5 as the thousands digit.

Choose 1 number to be at the ten thousands digit. We have 4 choices.

4 ways.

Now, if 5 is at the ten thousands digit, we now apply casework on the placement of 4.

4 is the unit digit.

We can use the same argument from above (since 5 is basically nonexistant).

1 way.

4 is the tens digit.

Same argument. Pick a number at the unit digit

3 ways

4 is the hundreds digit.

Same argument. Pick a number at the thousands digit.

3 ways.

If 4 is at the thousands digit we now apply casework on the placement of 3.

3 is the unit digit.

1 way.

3 is the tens digit.

Any combination of $1, 2$ work.

2 ways.

3 is the hundreds digit.

Any combination of $1, 2$ work.

2 ways.

We counted all the cases! (This only looks long; however, you can easily see the pattern and solve this quickly!)

So we add all our cases we found and have: $\boxed{27}$

13. The number 7 can be expressed as a sum of primes in the following 3 ways: $7, 5 + 2, 3 + 2 + 2$. In how many

ways can the number 19 be expressed as a sum of primes.

Be organized! Especially with the 2. Notice that when you sum with odd number of numbers, we need even numbers of 2, when you sum with even number of numbers, we need odd numbers of 2.

1 number: 19

2 numbers: $2 + 17$

3 numbers: $3 + 3 + 13, 3 + 5 + 11, 5 + 7 + 7$

4 numbers: $2 + 3 + 3 + 11, 2 + 3 + 7 + 7, 2 + 5 + 5 + 7, 2 + 2 + 2 + 13$

5 numbers: $3 + 3 + 3 + 3 + 7, 3 + 3 + 3 + 5 + 5, 2 + 2 + 3 + 5 + 7, 2 + 2 + 5 + 5 + 5, 2 + 2 + 2 + 2 + 11$

6 numbres: $2 + 3 + 3 + 3 + 3 + 5, 2 + 2 + 2 + 3 + 3 + 7, 2 + 2 + 2 + 3 + 5 + 5$

7 numbers: $2 + 2 + 3 + 3 + 3 + 3 + 3, 2 + 2 + 2 + 2 + 3 + 3 + 5, 2 + 2 + 2 + 2 + 2 + 2 + 13$

8 numbers: $2 + 2 + 2 + 2 + 2 + 3 + 3 + 3, 2 + 2 + 2 + 2 + 2 + 2 + 2 + 5$

9 numbers: $2 + 2 + 2 + 2 + 2 + 2 + 2 + 3$

It is impossible to have 10 numbers or more since the minimum sum if $2 \times 10 = 20 > 19$.

We count all of them and we get: $\boxed{23}$

Another solution:

We can use a sort-of recursion to tackle this problem.

Let $f(n)$ be such that $f(n)$ be the number of ways to write n as sum of primes.

Let $g(n)$ be such that $g(n)$ be the number of ways to write n as sum of primes without 2.

It is not hard to see that $f(n) = f(n-2) + g(n)$. Indeed, $f(n-2)$ represents the case when there is a 2.

$f(19) = f(17) + g(19)$.

$g(19)$ is not too hard to calculate, $(3 + 3 + 3 + 3 + 7, 3 + 3 + 3 + 5 + 5, 3 + 3 + 13, 3 + 5 + 11, 5 + 7 + 7, 19)$ and we have 6.

$f(19) = f(17) + 6$.

$f(17) = f(15) + g(17)$.

$g(17) = (3 + 3 + 3 + 3 + 5, 3 + 3 + 11, 3 + 7 + 7, 5 + 5 + 7, 17)$. We have 5.

$f(17) = f(15) + 5$. $f(19) = f(15) + 11$.

$f(15) = f(13) + g(15)$.

$g(15) = (3 + 3 + 3 + 3 + 3, 5 + 5 + 5, 3 + 5 + 7)$. We have 3.

$f(15) = f(13) + 3$. $f(19) = f(13) + 14$

$f(13) = f(11) + g(13)$.

$g(13) = (3 + 3 + 7, 3 + 5 + 5, 13)$. We have 3.

$f(13) = f(11) + 3$. $f(19) = f(11) + 17$.

$f(11) = f(9) + g(11)$.

$g(11) = (3 + 3 + 5, 11)$. We have 2.

$f(11) = f(9) + 2$. $f(19) = f(9) + 19$.

$f(9) = f(7) + g(9)$.

$g(9) = (3 + 3 + 3)$. 1 only.

$f(19) = f(7) + 20$. However, the problem has given us the value of $f(7) = 3$. Thus, we have $f(19) = 20 + 3 = \boxed{23}$.

14. The sum of the opposite faces of a traditional die is always 7. One face of one tradional die is selected at random and is glued to the face of another tradional die such that the two glued face have the same number. YOu roll the glued dice so that from above you can see one face of each die. If the sum of the two visible faces from above is 8, what is the probability that the two faces with number 1 are glued together? Express your answer as common fraction in lowest terms.

We explore cases on the number that was rolled on the left dice.

Case 1: The roll was 1.

It is impossible to get a roll of 8.

Case 2 The roll was 2.

5 is at the bottom. For the other dice, 6 on top and 1 on bottom. A possible number that can be glued together is either 3 or 4. For each case, each dice can be reflected so there are $2^2 = 4$ ways. We have 8 ways so far and none of them have a 1 glued.

Case 3 The roll was 3.

4 is at the bottom. For the other dice, 5 is on top and 2 is at the bottom. A possible number that can be glued together is 1 or 6. For each case, each dice can be reflected so there are $2^2 = 4$ ways. We have 8 ways with 4 of them having 1 glued together.

Case 4 The roll was 4.

3 is at the bottom. Same for the other dice. $1, 2, 5, 6$ are possible to be glued. Using logic from above, 4 ways for each case. We have 16 ways with 4 of them having 1 glued together.

Case 5 The roll was 5.

Identital with Case 3. 8 with 4 desired.

Case 6 The roll was 6.

Identical with case 2. 8.

We count all the total and the all the desired. The probability is $\frac{desired}{total} = \frac{12}{48} = \boxed{\frac{1}{4}}$

15 Let P be a point inside square $ABCD$. Given that $OA = 1, OB = 4$, and $OC = 5$, find the area of the square $ABCD$.

No diagram, so draw on your own! Draw big diagram!

Extend O such that it meets AB perpendicularly at E, meets BC perpendicularly at F and meets AD perpendicularly at J.

Let $AB = x, BF = y$. Through pythagoras, $BF^2 + FO^2 = BO^2$. $OF^2 = 16 - y^2$.

Through pythagoras, $CO^2 = CF^2 + OF^2$. $25 = 16 - y^2 + (x - y)^2$. Solving for y, we get $y = \frac{x^2 - 9}{2x}$.

$AE = BJ = EF - FO = x - \sqrt{16 - y^2}$

$AE^2 + EO^2 = AO^2$

$x^2 - 2x\sqrt{16 - (\frac{x^2-9}{2x})^2} + 16 - y^2 + y^2 = 1$

$x^2 + 15 = 2x\sqrt{16 - (\frac{x^2-9}{2x})^2}$

$\frac{x^2+15}{2x} = \sqrt{16 - (\frac{x^2-9}{2x})^2}$

$(\frac{x^2+15}{2x})^2 = 16 - (\frac{x^2-9}{2x})^2$

$(x^2 + 15)^2 + (x^2 - 9)^2 = 64x^2$

$2x^2 - 52x + 306 = 0$

$2(x^2 - 17)(x^2 - 9)$.

Thus, $x = 3$ or $x = \sqrt{17}$. However, when $x = 3$, that would mean that BOC would be a right triangle (do you see why?)and that does not make sense! Thus, we know that $x^2 = \boxed{17}$

Challenge: Where would O be such that $x^2 = 9$?

Alternate solution (this solution involves trigonometry):

Let $AB = x$.

Let $\angle OBA = \alpha$. Clearly $\angle CBO = 90 - \alpha$. Now apply law of cosine on the two triangles OBA and CBO.

For OBA, we have:

$16 + x^2 - 8x\cos(\alpha) = 1$.

$\cos(\alpha) = \frac{x^2+15}{8x}$

For CBO,

$16 + x^2 - 8x\cos(90 - \alpha) = 25$. Now, notice that $\cos(90 - \alpha) = \sin(\alpha)$. To show this, explore a right triangle with one of the angle α!

$\sin(\alpha) = \frac{x^2-9}{8x}$

From the definition of sin and cos (opposite/hypotenuse, adjacent/hypotenuse, respectively), we can easily see that $\sin^2(\alpha) + \cos^2(\alpha) = 1$.

Thus, we have $(\frac{x^2+15}{8x})^2+(\frac{x^2-9}{8x})^2=1$.

$2x^4+12x^2+306=64x^2$ and the rest is what was done in the previous solution.