Math Challengers Provincial 2016 Blitz and Bull's-eye rounds.

Solutions proposed by Sean Wang from Point Grey

Blitz

1. What is the sum of all single digit primes?

Solution: Recall that a prime number only has two distinct positive factors: 1 and itself. The single digit primes are $2, 3, 5, 7$, and thus their sum is $2 + 3 + 5 + 7 = \boxed{17}$

2. What is the largest common divisor of 16 and 20?

Solution: The positive factors of 16 are $1, 2, 4, 8, 16$, and the positive factors of 20 are $1, 2, 4, 5, 10, 20$. The largest number common to both lists is $\boxed{4}$

3. What is the smallest common multiple of 16 and 20?

Solution: We list out some of the multiples of 16 and 20.

Multiples of 16: $16, 32, 48, 64, 80, 96 \cdots$ Multiples of 20: $20, 40, 60, 80, 100, 120 \cdots$

The smallest number common to both lists is $\boxed{80}$

4. One angle of a convex pentagon is a right angle. What is the sum (in degrees) of the other four angles?

Solution: If you draw two non-intersecting diagonals of a convex pentagon, you will see that the pentagon is split into three triangles. Each triangle has angles that sum to $180°$, so a pentagon has angles that sum to $540°$. If one angle is $90°$, then the other four angles must sum to $540° - 90° = \boxed{450}^{\circ}$

5. You bought 5 pears and 4 apples. The cost of the apples was \$0.49 each, and the total cost of all apples and pears was \$5.41. What was the cost of each pear? Provide your answer in \$ correct to 2 decimal places.

Solution: The cost of 4 apples is $4 * \$0.49 = \$1.96$, so the cost of the 5 pears is $\$5.41 - \$1.96 = \$3.45$. Thus each pear costs $\frac{\$3.45}{5} = \boxed{\$0.69}$

6. You throw a (6-sided) die twice. What is the probability that on the second throw you get a larger number than you got on the first throw? Provide your answer as fraction in simplest form.

Solution: If your first throw is a 1, then your second throw should be a 2, 3, 4, 5, or 6. There are 5 possibilities for this case.

If your first throw is a 2, then your second throw should be a 3, 4, 5, or 6. There are 4 possibilities for this case.

If your first throw is a 3, then your second throw should be a 4, 5, or 6. There are 3 possibilities for this case.

If your first throw is a 4, then your second throw should be a 5 or 6. There are 2 possibilities for this case.

If your first throw is a 5, then your second throw should be a 6. There is 1 possibility for this case.

If your first throw is a 6, then your second throw cannot be greater than a 6. There are no posibilities for this case.

The total number of desired possibilites is $5+4+3+2+1=15$, and the number of possible outcomes is $6^2=36$. Our probability is $\frac{15}{36}=\boxed{\dfrac{5}{12}}$

7. If you spend on average 70 seconds on each of the 26 Blitz questions, how many minutes do you spend in total on Blitz? Round your answer to the nearest integer.

Solution: You spend an average of $\frac{\frac{70 \text{ seconds}}{60 \frac{\text{seconds}}{\text{minute}}}}{\text{question}}=\frac{7 \text{ minutes}}{6 \text{ questions}}$. There are 26 questions, so your total time is 26 questions $*\frac{7 \text{ minutes}}{6 \text{ questions}}=\frac{91 \text{ minutes}}{3}$. Rounding this to the nearest integer, we get $\boxed{30}$ minutes.

8. a and b are the solutions of $x^2-x-2=0$. Find the value of $\frac{ab}{a+b}$.

Solution 1:

By Vieta's formulas, $ab=-2$ and $a+b=1$. Thus $\frac{ab}{a+b}=\boxed{-2}$.

Solution 2:

Factor the equation to $(x-2)(x+1)=0$. Thus $a=2$ and $b=-1$ (in either order; it doesn't matter). The solution above follows.

9. What percent of 2016 is the number 252? Provide your answer as decimal correct to 1 decimal place.

Solution: $2016=2^5 3^2 7$

$252=2^2 3^2 7$

Thus, the answer is $\frac{2^2 3^2 7}{2^5 3^2 7}100\%=\frac{100\%}{8}=\boxed{12.5}\%$

10. Braille characters consist of minimum of one and maximum of six dots in the six positions, [as in the figure below]. How many Braille characters in total can be formed?

Solution: Each dot can be either on or off. There are 6 dots, so there are $2^6=64$ positions. However, the Braille character with all dots off is not valid, so we subtract one to get $64-1=\boxed{63}$

11. How many Braille characters consist of exactly four dots, of which at least is in the middle row? The figure for Question 10 shows one such character.

Solution: Choose any 4 dots. $\binom{6}{2} = \frac{6*5}{2*1} = 15$. However, we have ignored the restriction of at least one is in the middle row. One of our 15 characters is invalid. Thus, our answer is $15 - 1 = \boxed{14}$

12. Ann read a book of 600 pages. She read page 1 and then every second page after page 1 (i.e. she read pages $1, 3, 5, \cdots$); then she read every third page starting at page 1 (i.e. she read pages $1, 4, 7, \cdots$); and, then every fifth page starting at page 1 (i.e. she read pages $1, 6, 11, \cdots$). When she finished reading, how many pages she did not read?

Solution: $600 = 2^3 3 5^2$

$\phi(600) = 600(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5}) = \boxed{160}$

13. In Question 12, what is the first page after page 152 that Ann did not read?

Solution: Subtract 1 from all page numbers. We are looking for the smallest integer greater than 151 which is not divisible by 2, 3, or 5. 152, 153, 154, 155, and 156 are all read by Ann. However, 157 isn't, so our answer is $157 + 1 = \boxed{158}$

14. What is the value of $1 + 2 + 3 + \cdots + 38 + 39 + 40 + 39 + 38 + \cdots + 3 + 2 + 1$?

Solution: Group the integers $1 - 39$, 40, and $39 - 1$.

Consider the formula for the first n positive integers: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

The sum of the first 39 positive integers is $\frac{39(40)}{2}$. Because we have two of these sums, the answer is $\frac{2(39)(40)}{2} + 40 = \boxed{1600}$

15. The mean of 9 and N is the same as the mean of 10, 11, and N. What is the value of N?

Solution: We have the equation $\frac{9+N}{2} = \frac{10+11+N}{3}$. Multiplying both sides of the equation by 6, we get $27 + 3N = 42 + 2N$. Subtracting $27 + 2N$ from both sides gives us $N = \boxed{15}$

16. Some Grade 8 students and some grade 9 students work in the biology lab. If each Grade 8 student is given 2 samples to analyze, and each Grade 9 student is given 3 samples to analyze, then, in totalthey are given 39 samples. If, instead, each Grade 8 student is given 4 samples to analyze, and each Grade 9 student is given 5 samples to analyze, then in total they are given 69 samples. How many Grade 9 students work in the lab?

Solution: Let the number of Grade 8 students be x and the number of Grade 9 students be y.

We have the system of linear equations.

$2x + 3y = 39$

$4x + 5y = 69.$

Multiply the first equation by 2.

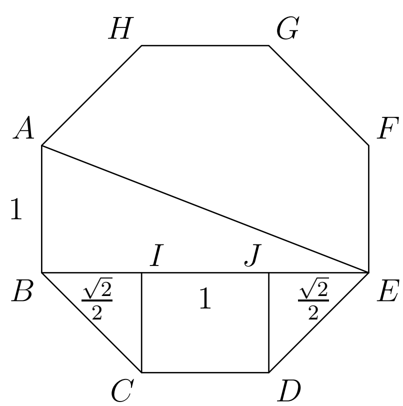
$4x + 6y = 78.$

Then subtract the second equation from that.

$y = \boxed{9}$

17. The figure below is of a regular octagon with side 1. What is the value of the square of the length of the diagonal AE? Express your answer as $i + j\sqrt{k}$, where i, j, and k are integers?

Solution:



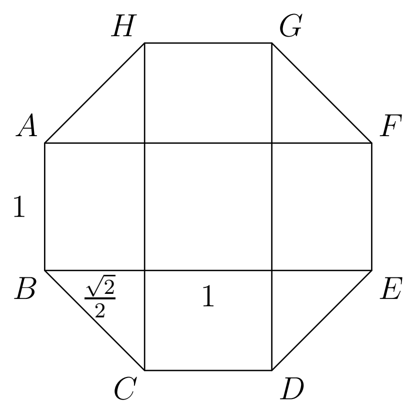
Connect B and E. Drop perpendiculars from C and D to BE. Call the intersection points I and J respectively. Because $\triangle BIC$ is right and isosceles (by symmetry), $BI = \frac{\sqrt{2}}{2}$. By the same reasoning, $JE = \frac{\sqrt{2}}{2}$. Obviously, $IJ = 1$, so $BE = 1 + \sqrt{2}$. Now, by using the Pythagorean theorem,
$AE^2 = AB^2 + BE^2 = 1^2 + (1 + \sqrt{2})^2 = 1 + 1 + 2\sqrt{2} + 2 = \boxed{4 + 2\sqrt{2}}$

18. In Question 17, how many diagonals are of the shortest length?

Solution: As labeled in question 17, diagonals that are of the shortest length have endpoints with letters that are "two apart, with H looping to A". In this case, $AC, BD, CE, DF, EG, FH, GA, HB$ are the $\boxed{8}$ diagonals of the shortest length. Notice that AB is not a diagonal.

19. In Question 17, what is the area of the octagon? Express your answer as $i + j\sqrt{k}$, where i, j, and k are integers?

Solution:



Connect CH, DG, BE, and AF. The octagon is split into 4 identical isosceles right triangles with side length $\frac{\sqrt{2}}{2}$, 4 identical rectangles with side lengths 1 and $\frac{\sqrt{2}}{2}$, and one square with side length 1.

The total area is $4\frac{(\frac{\sqrt{2}}{2})^2}{2} + 4\frac{\sqrt{2}}{2} + 1 = \boxed{2 + 2\sqrt{2}}$

20. Take 2016, divide it by 3, and round the result down to the nearest integer. Keep repeating by dividing the result by 3 and round down to the nearest integer. How many times you need to do this until reaching the result of zero?

Solution: The notation $\lfloor x \rfloor$ means the greatest integer less than or equal to x.

$\lfloor \frac{2016}{3} \rfloor = 672$

$\lfloor \frac{672}{3} \rfloor = 224$

$\lfloor \frac{224}{3} \rfloor = 74$

$\lfloor \frac{74}{3} \rfloor = 24$

$\lfloor \frac{24}{3} \rfloor = 8$

$\lfloor \frac{8}{3} \rfloor = 2$

$\lfloor \frac{2}{3} \rfloor = 0$

We applied this division $\boxed{7}$ times.

21. What number is the sum of all odd factors of 2016?

Solution: $2016 = 2^5 3^2 7$

An odd factor of 2016 must have no powers of two. Thus, we wish to find the sum of all factors of $3^2 7$.

The answer is $(3^0 + 3^1 + 3^2)(7^0 + 7^1) = \boxed{104}$

22. How many 11 digit numbers are there using the digit 3 five times and the digit 4 six times?

Solution: Suppose that each of the 11 digits is denoted as a blank as follows: _ _ _ _ _ _ _ _ _ _ _

There are $\binom{11}{5} = \frac{11*10*9*8*7}{5*4*3*2*1} = \boxed{462}$ to arrange the 3's. After that, the position of the 4's is fixed.

23. The hiking club of 8 girls and 7 boys prepares for a weekend activity and forms a planning committee of 3, of which no more than 2 are boys, and no more than 2 are girls. In how many ways can it be done?

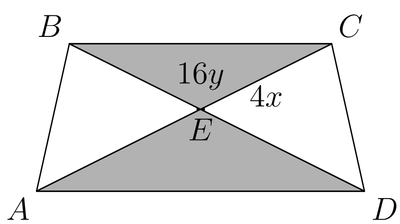
Solution 1 (Complementary Counting): We can find the number of ways to choose three people without restriction ($\binom{8+7}{3}$), then subtract the possibilities in which all 3 are girls ($\binom{8}{3}$) and the possibilities in which all 3 are boys ($\binom{7}{3}$).

Our answer is $\binom{15}{3} - \binom{8}{3} - \binom{7}{3} = \frac{15*14*13}{3*2*1} - \frac{8*7*6}{3*2*1} - \frac{7*6*5}{3*2*1} = 455 - 56 - 35 = \boxed{364}$

Solution 2 (Direct Counting): If two girls and one boy are chosen, there are $7\binom{8}{2}$ possibilities. If one girl and two boys are chosen, there are $8\binom{7}{2}$ possibilities.

Our answer is $7\binom{8}{2} + 8\binom{7}{2} = 7\frac{8*7}{2} + 8\frac{7*6}{2} = 196 + 168 = \boxed{364}$

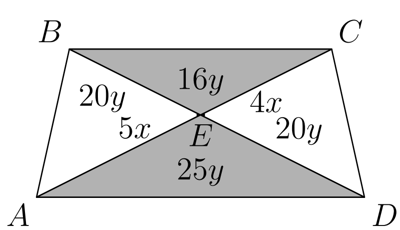
24. The picture below shows a trapezoid and its two diagonals. The two parallel sides of the trapezoid have length 4cm and 5cm, and the trapezoid has area 9cm^2. What is the area (in cm^2) of the shaded region? Express your answer as fraction in lowest terms.



Solution: Label the intersection of the diagonals E. Suppose that $CE = 4x$.

Because the diagonals act as a transversal, by AA similarity, $\triangle BEC \sim \triangle DEA$. Thus $AE = 5x$. Call the area of $\triangle BEC = 16y$. Because $\triangle BEC$ and $\triangle BAC$ share the same height but have bases $4x$ and $5x$ respectively, the area of $\triangle BAC = 20y$, and by similar reasoning, the area of $\triangle CED = 20y$.

Using the same idea of same-altitude-different-bases, we can obtain that the area of $\triangle AED = 25y$ ($\triangle DCE$ has base of $4x$, and $\triangle AED$ has base of $5x$).



We are given that the total area $16y + 20y + 20y + 25y$ equals $9cm^2$, thus $y = \frac{1}{9}$. Our answer is $16y + 25y = \boxed{\frac{41}{9}} cm^2$

25. How many positive integers smaller than 2016 contain both the digits 1 and 2 in their decimal representation?

Solution: Case 1: The positive integer has 2 digits.

There are clearly 2 such positive integers: 12 and 21.

Case 2: The positive integer has 3 digits.

If the positive integer starts with 1, the remaining two digits must have at least one 2 in it. There are $10^2 - 9^2 = 19$ such positive integers.

Similarly, if the positive integer starts with 2, there are also 19 such positive integers.

If the positive integer doesn't start with 1 or 2, there are $7 * 2 = 14$ such positive integers (7 possible choices for the first digit, second and third digits must be 1 and 2 in some order).

Case 3: The positive integer has 4 digits.

Subcase 1: The positive integer has first digit 1.

We use complementary counting to handle this subcase. There are $10^3 - 9^3 = 271$ such positive integers that contain at least one 2 in their last three digits.

Subcase 2: The positive integer has first digit 2.

There aren't many such positive integers for this subcase, so we list them all out: 2001, 2010, 2011, 2012, 2013, 2014, 2015. There are 7 such positive integers.

Thus, the answer is $2 + 19 + 19 + 14 + 271 + 7 = \boxed{332}$

26. What common fraction with denominator less than 20 lies between $\frac{7}{10}$ and $\frac{5}{7}$?

Solution: $\frac{7}{10} = 0.7$

$\frac{5}{7} = 0.\overline{714285}$

We try to test denominators to find the desired common fraction. We can estimate this by testing positive integers less than 20 is just smaller than a positive integer when multiplied by 0.7.

Start with 2:

2: 1.4

3: 2.1

4: 2.8

5: 3.5

6: 4.2

7: 4.9 discard this because $\frac{5}{7} \not< \frac{5}{7}$

8: 5.6

...

15: 10.5

16: 11.2

17: 11.9

18: 12.6

19: 13.3

It seems that a common denominator of 17 would be our answer. Indeed $\boxed{\frac{12}{17}}$ is our desired answer.

NOTE: This isn't a completely rigorous proof, but given the restrictions, the given solution is practical.

Bull's Eye

1. Alicia and Beti were the only candidates in the election for Student Council President. Alicia got 56% of the votes, and Beti got the remaining 44%. Alicia got 144 votes more than Beti. How many votes did Alicia get?

Solution: Let the total number of votes be x.

We have the equation: $0.56x - 0.44x = 144$. Thus $0.12x = 144 \implies x = 1200 \implies 56\% * 1200 = \boxed{672}$

2. Find the sum of the roots of the equation $(x - 63)(x - 32) = 2016$.

Solution: Expanding the left hand side, we get $x^2 - 95x + 2016 = 2016 \implies x^2 - 95x = 0$. By Vieta's formulas, the answer is $\boxed{95}$

3. There are 5 bowls, which between them contain a total of 100 almonds. Bowl 1 and Bowl 2 contain a total of 55 almonds. Bowl 2 and Bowl 3 contain a total of 41 almonds. Bowl 3 and Bowl 4 contain a total of 36 almonds. Bowl 4 and Bowl 5 contain a total of 28 almonds. How many almonds are in Bowl 1?

Solution: For each i in (1, 2, 3, 4, 5), define a_i as the number of almonds in the i-th bowl.

We have the following equations:

$a_1 + a_2 + a_3 + a_4 + a_5 = 100$

$a_1 + a_2 = 55$

$a_2 + a_3 = 41$

$a_3 + a_4 = 36$

$a_4 + a_5 = 28.$

Add the third and fifth equations together to get $a_2 + a_3 + a_4 + a_5 = 69$, and subtracting this from the first equation gives us $a_1 = \boxed{31}$

4. When 2016 is divided by the positive integer M, the remainder is 1. When 2016 is divided by $M + 2$, the remainder is 3. What is the smallest possible value of M?

Solution: From the first sentence, we know that M is a factor of 2015, but not 1. From the second sentence, we know that $M + 2$ is a factor of 2013.

The first few factors of 2015, excluding 1, are $5, 13, 31 \cdots$.

Check: Does $5 + 2$ divide 2013? No.

Check: Does $13 + 2$ divide 2013? No.

Check: Does $31 + 2$ divide 2013? Yes. The answer is $\boxed{31}$

5. Evaluate $1+2+4+5+7+8+\cdots+55+56+58+59$ (the sum of integers from 1 to 59, with every third integer omitted).

Solution: Consider the formula for the first n positive integers: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

The sum of the first 60 positive integers is $\frac{60(61)}{2} = 1830$. We have overcounted the multiples of 3. The sum $3+6+9+\cdots+57+60 = 3(1+2+3+\cdots+29+30)$. The sum of the first 20 multiples of 3 is equal to 3 times the sum of the first 20 positive integers. Thus $3+6+9+\cdots+57+60 = \frac{3(20)(21)}{2} = 630$, so our answer is $1830 - 630 = \boxed{1200}$

6. What number has binary representation of 11111100000?

Solution: $11111111111_2 - 00000011111_2 = 2047 - 31 = \boxed{2016}$

7. A bowl contains 10 jelly beans, of which 3 are black and the remaining 7 are yellow. Alphonse eats the jelly beans one at a time, each time piking a jelly bean at random. What is the probability that the third jelly bean that he eats is the second black one that he eats? Express your answer as a common fraction in lowest terms.

Solution: Case 1: He picks a black jelly bean, then a yellow jelly bean.

The probability of this occuring is $\frac{3}{10} * \frac{7}{9} = \frac{7}{30}$.

Case 2: He picks a yellow jelly bean, then a black jelly bean.

The probability of this occuring is $\frac{7}{10} * \frac{3}{9} = \frac{7}{30}$.

The sum of those two probabilities is $\frac{7}{30} + \frac{7}{30} = \frac{7}{15}$. We multiply this by $\frac{2}{8} = \frac{1}{4}$ to account for the probability that he eats a black jelly bean on his third pick. Our answre is $\frac{7}{15} * \frac{1}{4} = \boxed{\frac{7}{60}}$

8. What is the largest prime that divides $30! + 31! + 32! + 33!$?

Solution: Factor out the 30! from the sum.

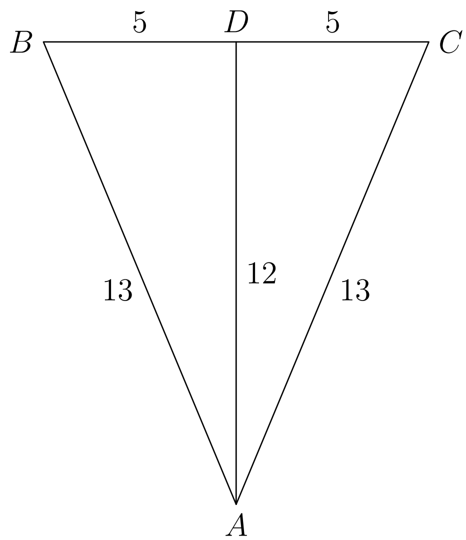
$30! + 31! + 32! + 33! = 30!(1 + 31 + 31 * 32 + 31 * 32 * 33) = 30!(1 + 31 + 992 + 33736) = 30!(33760)$

The prime factorization of 33760 is $2^5 5 * 211$. 211 is clearly bigger than the largest prime factor of 30! (which is 29), so the answer is $\boxed{211}$

9. What is the area of a triangle with sides 10, 13, and 13?

Solution 1: By Heron's formula, the semiperimeter is $s = \frac{13+13+10}{2} = 18$. Thus the area is $\sqrt{18(18-13)(18-13)(18-10)} = \sqrt{18 * 5 * 5 * 8} = \boxed{60}$

Solution 2:



Connect the midpoint of the side with length 10 to the vertex where the two sides with length 13 intersect. This line segment is the height of the triangle. The length of this line segment is $\sqrt{13^2 - \frac{10}{2}} = 12$. Thus, the area is $\frac{10*12}{2} = \boxed{60}$

10. The number of diagonals of a regular polygon is 6 times the number of the edges of the polygon. How many edges does the polygon have?

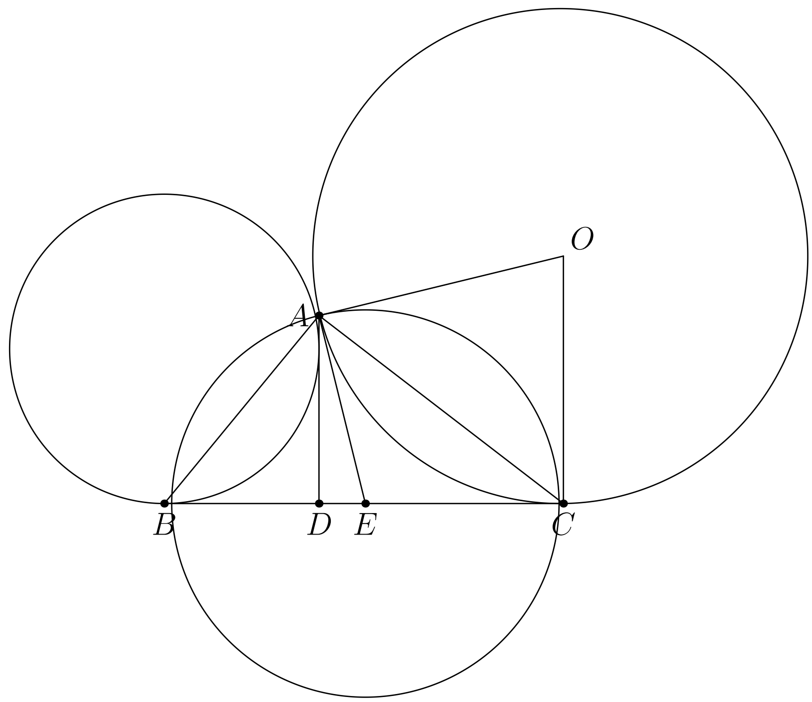
Solution: The number of diagonals in a regular polygon with n sides is $\frac{n(n-3)}{2}$.

Thus, we have the equation $\frac{n(n-3)}{2} = 6n \implies n(n-3) = 12n \implies n^2 - 15n = 0 \implies n = 0, 15$.

Since a regular polygon cannot have 0 sides, we discard that solution, so the polygon has $\boxed{15}$ edges.

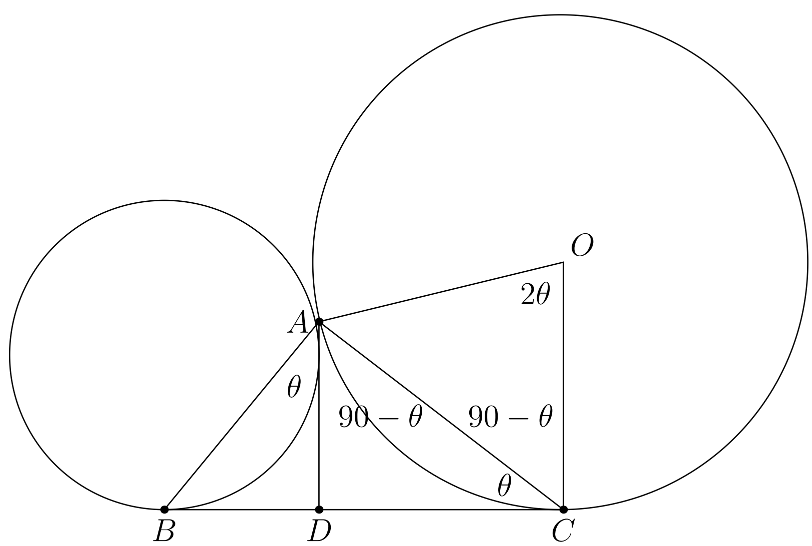
11. Two circles, a smaller one, and a larger one with center at O, are externally tangent at A. The circles are also tangent to the line BC where B and C are the tangent points. The line AD is perpendicular to BC and D is on BC. What is th ratio of $\angle BAD$ to $\angle AOC$? Express your answer as a common fraction in lowest terms.

Solution:



Lemma: $\angle BAC = 90°$

Proof: Draw the line tangent to the two circles at A. Call the intersection point between that line and BC as E. Because AE and BE are both tangent to the smaller circle, $AE = BE$. Because AE and CE are both tangent to the larger circle, $AE = CE$. By combining these equalities, we see that $AE = BE = CE$. Thus, BC is a diameter of the circumscribed circle of $\triangle BAC$. Because $\angle BAC$ subtends half of the circumscribed circle, $\angle BAC = 90°$.



Let $\angle AOC = 2\theta$. Then, $\angle ACO = \frac{180-2\theta}{2} = 90 - \theta \implies \angle ACD = \theta \implies \angle CAD = 90 - \theta \implies \angle BAD = \theta$ (angle chasing).

$\frac{\angle BAD}{\angle AOC} = \frac{\theta}{2\theta} = \boxed{\frac{1}{2}}$

12. In Question 11, $AB = 4$ and $AC = 5$. What is the value of AD? Express your answer as $\frac{m}{\sqrt{n}}$, where m is an integer and n is a prime number.

Solution: Because $\angle BAC = 90^\circ$ as shown in the above solution, the length of BC is $\sqrt{4^2 + 5^2} = \sqrt{41}$.

The area of $\triangle BAC$ can be written in two different ways: $\frac{AB*AC}{2}$ and $\frac{BC*AD}{2}$. The former equals $\frac{4*5}{2} = 10$, and the latter equals $\frac{AD\sqrt{41}}{2}$. Since they are equal, $\frac{AD\sqrt{41}}{2} = 10$, so $AD = \boxed{\frac{20}{\sqrt{41}}}$